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NASA Technical Paper 1219

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Volume I

Carl S. Bodley, A. Darrell Devers,
A. Colton Park, and Harold P. Frisch

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All measurement values are expressed in the International System of Units (SI) in accordance with NASA Policy Directive 2220.4, paragraph 4.

PREFACE

This document, prepared by the Dynamics and Loads Section, Martin Marietta Corporation, Denver Division, under Contract NAS5-11996, presents the results of a study for the purpose of developing a computer program system for dynamic simulation and stability analysis of passive and actively controlled spacecraft. The study was performed from May 1973 to April 1975 and was administered by the Goddard Space Flight Center, National Aeronautics and Space Administration, Greenbelt, Maryland, under the direction of Joseph P. Young.

Upon delivery, the computer program and associated documentation was checked in detail by Harold P. Frisch. This document incorporates both the original Martin work and the supplementary material prepared at the Goddard Space Flight Center.

The digital computer program, DISCOS (Dynamic Interaction Simulation of Controls and Structure), has been extensively annotated and tested on a range of problems that should have exposed nearly all theoretical errors and programming bugs.

From its inception in 1973, DISCOS has been designed to grow as new needs and more efficient computational techniques develop. This feature makes it impossible to define a final version. To circumvent this problem, the official release version will contain only those additions to the delivered program that enhance program documentation and user interface capability and correct programming errors.

Included in this version are more than 10,000 comment cards and a capability to routinely direct the computer to output on the line printer virtually all computation along with explanatory alphanumeric statements. A large percentage of the comment cards are in subroutines DEF1, DEF2, . . ., and DEF5. These subroutines are composed entirely of comment cards and provide the user with an area in the source file for keeping documentation current. In particular, subroutine DEF5 contains a narrative description of the program and its current capabilities.

For the uninitiated reader, it probably would be best to speed-read subroutine DEF5 to obtain a quick overview of the capabilities of the program and the solution techniques applied before reading this document.

The potential user should be aware of the fact that DISCOS is not intended for simple problems. It is primarily for problems which were heretofore intractable. Consequently, the theoretical basis for the program is highly advanced, and the computation algorithms are designed for the efficient processing of the equations associated with large multidegree-of-freedom systems.

As an aid to the user, the paper on which the derivation of the coupled flexible-body equations of motion is based and a paper that outlines the solution method and comments on results obtained from several DISCOS applications appear as reference material at the end of Volume I. Volume I contains all relevant theoretical work.

Volume II describes the DISCOS program and its support programs. The user is encouraged to refer to the comment cards provided in all DISCOS subroutines for additional descriptive information. The comment cards found in the DISCOS subroutines are intended to provide a link between the computer code and the theoretical equations provided in Volume I.

To effectively interface with the program, the user must be able to write the subroutines that will define all non-gyroscope forces and torques. The user is provided with a clean interface. When the load vector associated with the effects of springs, dampers, motors, gas jets, etc. is defined and properly stored in the computer memory, DISCOS will automatically transform it into the appropriate generalized form required by the formulation.

The inclusion of effects such as aerodynamic loading and thermodynamic deformation is more difficult. However, the methodology is analogous to that used for including gravity-gradient effects.

The methodology for including loads associated with springs, dampers, motors, gas jets, constraints, etc. is found in Volume II and in the comment cards of the appropriate subroutines referred to in Volume II.

DISCOS is probably the most powerful computational tool to date for the computer simulation of actively controlled coupled multiflexible-body systems. It is not an easy program to understand and effectively apply, but it is not intended for simple problems. The user is expected to have an extensive working knowledge of rigid- and flexible-body dynamics, finite-element techniques, numerical methods, and frequency-domain analysis.

A DIGITAL COMPUTER PROGRAM FOR THE DYNAMIC INTERACTION SIMULATION OF CONTROLS AND STRUCTURE (DISCOS) VOLUME I

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I. INTRODUCTION

Modern society has derived numerous and varied benefits from Earth-orbiting satellites. These benefits include prediction of extreme changes in weather, exploration of mineral and water resources, materials research, medical research and experimentation, solar-system studies, and increased communications and defense capabilities. The design and ultimate development of these satellites requires extensive analytical and experimental studies to ensure complete confidence in the overall ability of the total system to perform its required functions.

Two of the most important and potentially most difficult of these studies are the analysis of dynamic response and the prediction of stability characteristics. In recent years, the National Aeronautics and Space Administration (NASA) and members of the aerospace industry have expended much effort in analyzing these phenomena. Although these are worthy efforts, most have been somewhat limited in scope because of the extreme complexity of the problems. Factors that make these studies so complex are: (1) the imposition of a spin rate for either stabilization or artificial gravity; (2) the dynamic interaction of this spin rate with large flexible appendages; (3) the complex environmental loadings, including gravity gradient and solar radiation; and (4) the intricate control logic required for maintaining stability or for executing orbital maneuvers.

The authors at Martin Marietta Corporation acknowledge the assistance provided by Goddard Space Flight Center (GSFC) personnel. Harold Frisch and James Donohue contributed valuable technical comments and suggestions throughout the program. In particular, they

developed the basic approach to be used for data input, defined exactly what should be included in the transfer function and stability analysis portion of the program, and defined eight of the eleven demonstration problems. They provided the associated data that were used to verify both the nonlinear time response and linear transfer function and the stability analysis portions of the program. Raymond Welch provided the subroutine used to generate root-locus plots. Dr. William Case provided valuable advice on interfacing with NASTRAN output and generated the demonstration problem used to validate the interface subroutine (NASFOR).* During the early program development stage, Dr. James Mason offered significant advice on the need to compute internal forces at the interconnect points. Reginald Mitchell contributed invaluable advice on requirements for making the program compatible with the GSFC IBM 360/95 computer system. In addition, he supplied the contractor with a 360-compatible plot package, furnished the contractor with a self-authored subroutine for reading NASTRAN output, and was responsible for running all demonstration problems on the 360/95 computer. Finally, the authors acknowledge the encouragement and efforts of Joseph P. Young, Technical Monitor, who made numerous valuable comments and suggestions throughout the study.

A. Overview

The state-of-the-art dynamic response analysis of a system of connected bodies is currently restricted to topological systems of connected rigid bodies with (possibly) flexible terminal bodies. Because of the complex orbiting configuration and mechanical systems proposed for future space programs, the limitations of the current technology are severely restrictive. This document presents a more comprehensive formulation that is capable of considering any body of the total system as flexible and that is not restricted to a specific connection arrangement.

Applications of such methods and program systems are numerous and include simulation of the Space Shuttle payload deployment/retrieval mechanism, solar-panel-array deployment, antenna deployment, analysis of multispin satellites, and analysis of large, highly flexible satellites.

This approach provides a general-purpose modeling capability for dynamic simulation and stability analysis of passive and actively controlled spacecraft. In particular, the following items are considered:

- Time-domain solution of the nonlinear differential equations of motion that describe total or nominal response† of the complete spacecraft system idealized as a collection of interconnected flexible (or rigid) bodies
- Linearization of the governing differential equations by numerical means

*The NASFOR subroutine has been superseded by a special NASTRAN DMAP program and associated preprocessor program written by Harold P. Frisch.

†In certain cases, the total response of the dynamic system may be considered to be equilibrium state motion (nominal response) plus perturbation motion with respect to the equilibrium state.

- Time-domain solution of the linearized differential equations that describe the perturbation response of the complete spacecraft system about some predetermined (calculated or user-specified) nominal motion
- General frequency-domain stability analysis corresponding to the linearized spacecraft representation
- Provision for arbitrary (explicitly time-dependent) loadings and environmental interaction, such as gravity gradient and thermally induced deformations resulting from solar radiation

B. Description of the Physical System

The physical system undergoing analysis may be generally described as a cluster of contiguous, flexible structures (bodies) that comprise a mechanical system such as a spacecraft. The entire system (spacecraft) or portions thereof may be either spinning or nonspinning. Member bodies of the spacecraft are capable of undergoing large relative excursions such as those of appendage deployment or rotor/stator motions. The general system of bodies is, by its inherent nature, a feedback system in which inertial forces (such as those due to centrifugal and Coriolis acceleration) and the restoring and damping forces are motion-dependent. Also, the system may possess a control system in which certain position and rate errors are actively controlled through the use of reaction control jets, servomotors, or momentum wheels.

Bodies of the system may be interconnected by linear or nonlinear springs and dampers, by a mechanism that is a combination of gimbal and slider block, or by any combination of these. Also, any two bodies of the system may be free (one from the other) and possess six degrees of relative motion freedom. Several or all of the six degrees of relative motion freedom between two bodies may also be a prescribed function of time (including zero motion).

For further introduction of the physical system, consider an illustrative example, such as the system of bodies of figure 1, and let this example be the prototype for all subsequent discussion and development.

In figure 1, a nontopological tree configuration has been deliberately indicated. There are five hinges and four bodies, thus one closed path. Consecutive integer labels are used for body reference points, hinges, sensor points,* and momentum wheels. The numerical order in each of the four label sets may be random; however, it is understood that body 1 (which may be any body of the system) is associated with hinge 1.

For each body of the system, there is a body-fixed, right-handed reference frame, whose origin may be at the body's mass center or at some structural hard point on the body. (A body's elastic deformation is measured in its reference frame.)

*A "sensor point" is any point at which kinematic data must be obtained (e.g., where an attitude sensor is located).

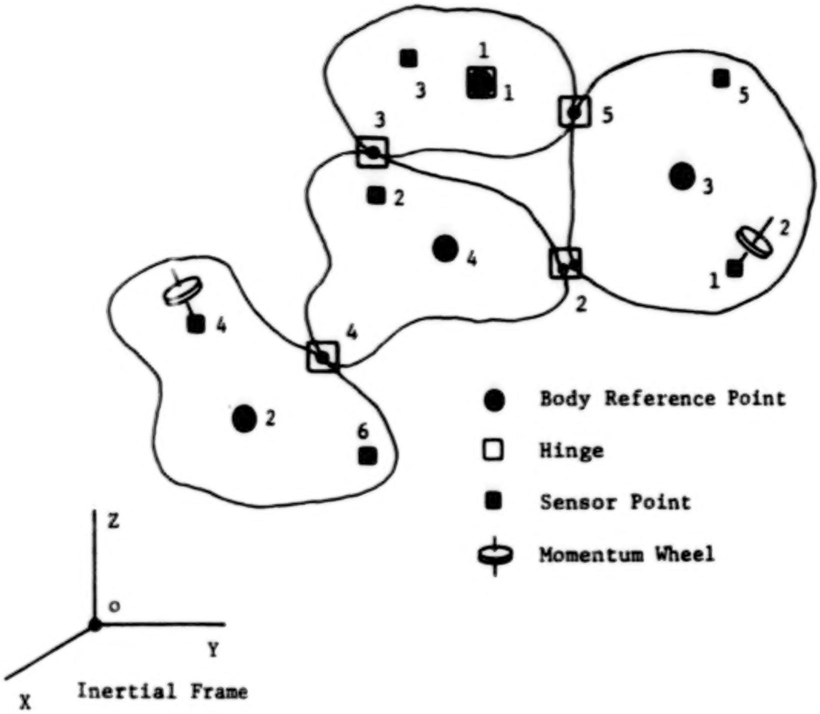


Figure 1. Labeling scheme for example system.

In this document, a hinge is defined as a pair of structural hard points (see figure 2) with a point situated on each of two contiguous bodies. In figure 2, point p and point q comprise a hinge. Clearly, a typical body may contain one or more hinge points, but a hinge may be associated with only two bodies. Hinge 1 is given special consideration. It is also a pair of points; but one of the pair is coincident with the reference point of body 1, and the other point of the pair is coincident with the inertial origin. Thus, motion "across the hinges" is used to represent system motion. The reference point of body 1 is located with respect to the inertial origin by an inertially referenced position vector. The attitude of the reference frame of body 1, with respect to the inertial frame, is represented by three Euler angles. Thus, six position/attitude coordinates are associated with hinge 1.

Each of the remaining hinges is considered in a manner somewhat similar to that of hinge 1. Referring to figure 2, note that there is an orthogonal reference frame attached to point p and another to point q. The triad of point p may have a "natural" (or undeformed) misalignment with respect to the triad of body point m, plus additional misalignment caused by elastic deformation. The same relationship is true concerning points n and q.

Now six relative position/attitude coordinates are associated with the hinge of points p and q. Point q is located from point p with a p-frame referenced position vector. The attitude of the q-frame with respect to the p-frame is represented by three Euler rotations. Thus, if NH is the number of system hinges, 6 × NH position coordinates are to be used in

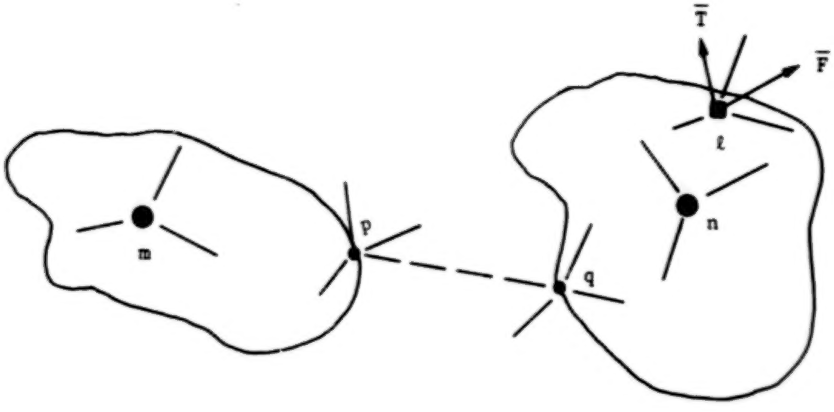


Figure 2. Typical contiguous bodies of the system.

conjunction with modal displacement coordinates for defining the system's position state. Note that only the time-variable position coordinates of the 6 × NH set of candidates are considered as state-vector elements. (The position coordinates whose rates are constrained to zero are not included; however, the position coordinates themselves need not be zero.)

The system of bodies usually has a number of so-called sensor points. A sensor point is defined as a structural hard point with an attached right-handed orthogonal reference frame that is used for a variety of purposes. A sensor point may be used to enable the program system to monitor the position, attitude, or the rates associated with a specific structural hard point. For example, a rate gyro, a star tracking device, or another motion/position sensing device is physically situated at a sensor point. Also, a sensor point is used as a point of application of a force or torque vector (see figure 2).

The system of bodies may contain built-in momentum wheels, of which some are constant speed wheels and others are variable speed. The variable speed momentum wheels are motor driven; the shaft torque results from a given control law. Each momentum wheel of the system must be associated with a sensor point because, for a general flexible body, the gyroscopic coupling is influenced by elastic motion.

As is indicated in figure 1, the system may be in a nontopological tree configuration. The methods employed in this development are such that closed-loop configurations (generally referred to as nontopological) may be considered. If every body of the N-Body system is rigid, there may be no closed loops because such a system has an indeterminate "load path." To accommodate closed loops, the system must contain sufficient structural flexibility (compliance), and therefore modal displacement coordinates, so that the kinematic equations of interconnection constraint are algebraically consistent.

The program development is such that none, several, or all bodies of the N-Body system may be flexible. The system may be "forced" by environmental factors such as gravity, gravity gradient, solar pressure, thermal gradient, and aerodynamic drag.

The computer program system described herein falls into several categories of capability:

- Synthesis and time-domain solution of the nonlinear differential equations of motion of the complete spacecraft system idealized as a collection of interconnected flexible (or rigid) bodies
- Linearization of the governing equations by numerical means
- Time-domain solutions of the linearized equations that describe perturbation response of the complete spacecraft system about some predetermined (calculated or user-specified) nominal motion
- General frequency-domain stability analysis corresponding to the linearized spacecraft representation

II. EQUATIONS OF STATE/TIME-DOMAIN SIMULATION

A. Introductory Discussion

The state equations* that govern dynamic response of a system of interconnected flexible bodies, which may be actively or passively controlled and which may be "forced" by environmental factors such as solar pressure, gravity gradient, aerodynamic drag, etc., are presented here in a concise summary form as:

$$\{\dot{U}\}_j = [m]_j^{-1}\left(\{G\}_j + [b]_j^T\{\lambda\}\right) \tag{II-1}$$

$$\{\dot{\xi}\}_j = [S_\xi]_j\{U\}_j \tag{II-2}$$

$$\{\dot{\beta}\} = \sum_j [B]_j\{U\}_j \tag{II-3}$$

$$\{\dot{\delta}\} = f\left(\{\beta\},\{\dot{\beta}\},\{\xi\},\{\dot{\xi}\},\{\delta\}\right) \tag{II-4}$$

subject to the constraint equations

$$\sum_j [b]_j\{U\}_j = \{\dot{\alpha}\} \tag{II-5}$$

*For the interested reader, Reference Paper I describes the development of these equations in more detail.

In equations II-1 through II-5 the index, j, ranges from 1 through the number of bodies of the system. Equations II-1 through II-4 represent n first order, nonlinear, ordinary differential equations whereas equation II-5 represents m additional conditions of kinematic constraint. Thus, the dimension of the state space for a given system of controlled bodies is (n-m). That is, n-m state variables are required for defining the configuration at any instant of time, t.

State variables of the configuration space include absolute velocities, $\{U\}_j$; modal displacements, $\{\xi\}_j$; position coordinates (both angular and cartesian position), $\{\beta\}$; and additional variables, $\{\delta\}$, that will subsequently be referred to as control variables. These variables are associated with the differential equations that define a given control law. However, they may reflect any other auxiliary differential equations that are necessary for defining the overall feedback system (for example, they may include thermal equilibrium states or other state variables necessary for a complete definition of the state-dependent environment)

The right-hand sides of equations II-1 through II-4 are functionally dependent on all the state variables and time, although the relationships may be termed only implicit at this point. Let it suffice that, in a way of introduction, a description of the nature of the governing equations II-1 through II-5 be given here and more explicit development and discussion follow in subsequent sections.

Equation II-1 represents the dynamic equilibrium equations for the typical j^{th} body of the system. They are of the form shown whether the body is treated as rigid or flexible. They state, in effect, that a deformation dependent mass matrix, $[m]_j$, postmultiplied by a vector of relative accelerations, $\{\dot{U}\}_j$, produces a vector of inertial forces that is balanced by all other state- and time-dependent forces, $\{G\}_j$, and interconnection constraint forces, $[b]_j^T\{\lambda\}$. The vector, $\{G\}_j$, includes inertial forces due to centrifugal and Coriolis acceleration, as well as elastic restoring forces, damping forces, control actuator forces, and so forth. The constraint forces, $[b]_j^T \{\lambda\}$, are necessary in order to satisfy the kinematic constraint equation (II-5); elements of the vector $\{\lambda\}$ are actually Lagrange multipliers, evaluated and used in the solution process.

Equation II-2 simply represents a selection transformation because the vector of modal velocities, $\{\dot{\xi}\}_j$, is a subvector of $\{U\}_j$. Equation II-3, used to develop $\{\dot{\beta}\}$, represents a kinematical transformation, transforming nonholonomic velocities to time derivatives of position coordinates. Finally, equation II-4 is an auxiliary differential equation that is user-defined and may be used to implement control dynamics and other feedback effects.

The constraint equation (II-5) represents kinematic conditions of a form similar to those of equation II-3. In either case, there is a velocity transformation. Equation II-5 might be termed an active set of kinematic conditions, and those of equation II-3, a passive set. The active set is used to calculate m of the dependent elements of the $\{U\}_j$ vectors in terms of

the remaining independent elements and the prescribed velocities, $\{\dot{\alpha}\}$, some of which may be zero and others user-defined functions of time. Thus, the constraint equations are of a general form because nonholonomic, rheonomic conditions may be so represented. If the $\{U\}_j$ vectors satisfy the required conditions of equation II-5, the position rates, $\{\dot{\beta}\}$, may be evaluated by the passive conditions of equation II-3, resulting in a kinematically consistent system.

Note that m equations of constraint are represented by equation II-5. There are also m Lagrange multipliers in the vector, $\{\lambda\}$. In studies of dynamic systems, the Lagrange multipliers and the dependent velocities and accelerations are often entirely eliminated from the governing equations. Such is not the case in this development in which Lagrange multipliers are involved in the equations for two reasons: (1) to monitor the multipliers as a function of system motion, as they are interconnection forces and torques, and (2) for numerical implementation, it is convenient to calculate and use the $\{\lambda\}$ vector in equation II-1. The Lagrange multipliers are calculated by differentiating equation II-5 and combining that result with equation II-1 giving:

$$\{\lambda\} = \left(\sum_j [b]_j\,[m]_j^{-1}\,[b]_j^T\right)^{-1} \left[\{\ddot{\alpha}\} - \sum_j \left([\dot{b}]_j\,\{U\}_j + [b]_j\,[m]_j^{-1}\,\{G\}_j\right)\right] \qquad \text{(II-6)}$$

Note the following functional dependencies:

$$[b]_j = f\left(\{\beta\}_j, \{\xi\}_j\right) \qquad \text{(II-7)}$$

$$[B]_j = f\left(\{\beta\}_j, \{\xi\}_j\right) \qquad \text{(II-8)}$$

thus,

$$\{\dot{\beta}\} = f\left(\{\beta\}, \{\xi\}, \{U\}\right) \qquad \text{(II-9)}$$

$$\{\dot{\xi}\}_j = f\left(\{U\}_j\right) \qquad \text{(II-10)}$$

$$\{\dot{\delta}\} = f\left(\{\beta\}, \{\dot{\beta}\}, \{\xi\}, \{\dot{\xi}\}, \{\delta\}\,;\, t\right) \quad \text{(II-11)}$$

$$\{G\}_j = f\left(\{\xi\}, \{U\}, \{\delta\}\,;\, t\right) \quad \text{(II-12)}$$

$$[m]_j = f\left(\{\xi\}_j\right) \quad \text{(II-13)}$$

$$[\dot{b}]_j = f\left(\{\beta\}, \{\dot{\beta}\}, \{\xi\}, \{\dot{\xi}\}\right) \quad \text{(II-14)}$$

thus,

$$\{\lambda\} = f\left(\{\xi\}, \{\beta\}, \{U\}, \{\dot{\xi}\}, \{\dot{\beta}\}, \{\dot{\delta}\}\,;\, t\right) \quad \text{(II-15)}$$

and

$$\{\dot{U}\} = f\left(\{\xi\}, \{\beta\}, \{U\}, \{\dot{\xi}\}, \{\dot{\beta}\}, \{\dot{\delta}\}\,;\, t\right) \quad \text{(II-16)}$$

where, in the foregoing notation, the elements of the matrices/vectors on the left are functions of the elements of the vectors on the right. The chronology of evaluations indicated must be followed in the solution process.

The differential equations of motion for the system are therefore of the general form:

$$\dot{y}_i = f(y_1, y_2, \ldots, y_{n-m}\,;\, t) \quad \text{(II-17)}$$

and the state vector and its time derivative are arranged as follows:

$$\{y\} = \begin{bmatrix} \{U\}_1 \\ \{U\}_2 \\ \cdot \\ \cdot \\ \{U\}_{NB} \\ \{\xi\}_1 \\ \{\xi\}_2 \\ \cdot \\ \cdot \\ \{\xi\}_{NB} \\ \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \beta_{N\beta} \\ \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \delta_{N\delta} \end{bmatrix} \qquad \{\dot{y}\} = \begin{bmatrix} \{\dot{U}\}_1 \\ \{\dot{U}\}_2 \\ \cdot \\ \cdot \\ \{\dot{U}\}_{NB} \\ \{\dot{\xi}\}_1 \\ \{\dot{\xi}\}_2 \\ \cdot \\ \cdot \\ \{\dot{\xi}\}_{NB} \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \cdot \\ \cdot \\ \dot{\beta}_{N\beta} \\ \dot{\delta}_1 \\ \dot{\delta}_2 \\ \cdot \\ \cdot \\ \dot{\delta}_{N\delta} \end{bmatrix}$$

where NB is the total number of bodies of the system, $N\beta$ is the total number of position coordinates necessary for orienting the system, and $N\delta$ is the total number of auxiliary (control) differential equations required.

Now, if the $\{y\}$ vector is known (numerically) from prescribed initial conditions or from numerical integration of $\{\dot{y}\}$, the primary task of the solution process is to numerically establish the $\{\dot{y}\}$ vector. The $\{\dot{y}\}$ vector is numerically (step by step) integrated to produce an incremented $\{y\}$ vector, and thus a sequence of time-point solutions.

To summarize, a narrative description of the steps (numerical evaluations) necessary for producing $\{\dot{y}\}$ given $\{y\}$, follows.

The matrices, $[B]_j$ and $[b]_j$, are kinematic coefficients that depend on position and modal displacement variables and are evaluated as the first step.

Now, if available numerical techniques (also computer software and hardware) were absolutely accurate, the $\{U\}_j$ vectors resulting from numerical integration of the $\{\dot{U}\}_j$ vectors would satisfy the constraint equation II-5. Because this is not the case, the second step of the solution process is to calculate the dependent elements of the $\{U\}_j$ vectors by using equation II-5. In fact, because of anticipated numerical inaccuracies, only the independent elements of the $\{\dot{U}\}_j$ vectors are obtained by numerical integration. Only n-m "integrators" are involved in the solution process even though all of the elements of the $\{\dot{U}\}_j$ vectors are numerically evaluated (by use of equation II-1); good numerical resolution is found in the independent $\{\dot{U}\}_j$ elements because the Lagrange multipliers $\{\lambda\}$ were used.

A kinematically consistent system results from satisfying equation II-5. The $\{U\}_j$ vectors may now be used with the selection and kinematic transformations as indicated by equations II-2 and II-3 to numerically produce all the modal velocities, $\{\dot{\xi}\}_j$, and position coordinate rates, $\{\dot{\beta}\}$, completing the third step of the process.

Sufficient calculation has been completed to this point to evaluate the control variable rates according to equation II-4, producing $\{\dot{\delta}\}$. During the process of calculating the $\{\dot{\delta}\}$ vector, all of the required control actuator torques (or forces) are calculated because sufficient numerical information is available. All of the constituents of the torques/force vectors, $\{G\}_j$, are now available, and $\{G\}_j$, $[m]_j$, and $[\dot{b}]_j$ are therefore numerically evaluated, which completes the fourth step of the process. (Refer to the functional expressions of equations II-11 through II-14.)

With reference to equation II-6, note that there is now sufficient numerical information to evaluate $\{\lambda\}$, which is then used in equation II-1 to calculate the $\{\dot{U}\}_j$, completing the fifth and final step of the process.

Note that, in the above discussions, the solution process may be carried out through completion, providing the state vector is numerically known. At any step of a simulation, the $\{y\}$ vector is known, of course, as the result of numerical integration. The initial state vector is another matter. It is difficult, if not impossible, for a user to prescribe $\{U\}_j$ vectors that are kinematically consistent with the conditions of equation II-5; also, the nonholonomic velocities of $\{U\}_j$, when considered as a complete set, are of a somewhat abstract nature. The user is in a much better position to prescribe initial values of $\{\dot{\beta}\}$ and $\{\dot{\xi}\}$ – the initial velocities that are physically meaningful to him. Thus, to initiate the simulation (that is, to create an initial state vector from information the user is in a position to prescribe) the following preliminary steps must be taken.

The user must prescribe initial values of the $\{\xi\}_j$, $\{\dot{\xi}\}_j$, $\{\beta\}$, $\{\dot{\beta}\}$, and $\{\delta\}$ vectors and the variable speed momentum-wheel spin velocities, $\dot{\theta}$. Now, in that the prescribed position rates, $\{\dot{\alpha}\}$, are explicitly dependent on time and are always available, kinematic equations II-3 and II-5 may be used together to establish initial values of all $\{U\}_j$. The question inevitably arises: Are the number of equations represented by II-3 and II-5 sufficient to solve for the elements of the $\{U\}_j$? Consider the typical $\{U\}_j$ vector. Note that there are six reference-frame velocities in each $\{U\}_j$ (namely, ω_x, ω_y, ω_z, u, v, and w). Six relative velocities are also associated with each hinge. Now, if the system is a topological tree configuration, equations II-3 and II-5 comprise exactly the required number of equations to establish the reference-frame velocities; that is, there are as many hinge points as there are bodies, and, even if every body were rigid, the system would be determinate. In this case, the initial sets of six reference-frame velocities are computed by equations II-3 and II-5; the prescribed initial $\{\xi\}$ vectors and momentum-wheel spin velocities are simply placed in the appropriate $\{U\}_j$ vectors, and the initial state vector is thus defined.

If the system is not a topological tree configuration, then there are more equations (II-3 and II-5) to be satisfied than there are reference frame velocities. (In other words, there are more hinges than bodies.) In this case, elements of the $\{\dot{\xi}\}_j$ vectors must take on the responsibility of helping to satisfy the kinematic conditions. For each hinge in excess of the number of system bodies, there must be at least six deformation modes, represented by ξ coordinates, and they must be distributed throughout the system in such a way that the kinematic conditions of equation II-5 are independent. Clearly then, when there are more hinges than bodies (nontopological tree), one or more of the bodies must be flexible for the system to be determinate. When the configuration is nontopological, the user will specify initial values for all of the $\dot{\xi}$, but he must acknowledge that they are not all independent and that the dependent ones (automatically determined by the program) are calculated and replace the values that he has specified.

From these considerations, note that the initial state vector is created by the program from information that is user-supplied and that is physically meaningful to him. In the event of a nontopological tree configuration, the user's only concern in regard to initial conditions is whether he has supplied an adequate description of system flexibility for the system's kinematical equations to be determinate.

B. Derivation of Equations of Dynamic Equilibrium

The differential equations of motion and auxiliary equations that characterize a physical system may take any one of several equivalent forms. Equivalent form means that the same physical system can be characterized by more than one set of mathematical variables; in any case, the number of variables must be the same. For example, the motion equations for a rigid body could be derived by using Lagrange's equations (resulting in six second-order equations), or the Newton-Euler equations could be used when translational motion is represented by three second-order equations, whereas rotational motion is represented by

six first-order equations (three moment-momentum equations and three attitude equations). In each case, there are twelve state variables.

There are a variety of alternative methods of analytical dynamics that one may select from to develop the final (programmable) equation format. A timely and valuable commentary accompanies the comprehensive comparative evaluation of these methods in a recent report by Likens (Reference 1). The basis for this development is effectively included in his discussion.

The intent here is not to highlight any particular method of analytical dynamics as being superior to the others. Clearly, the methods are all equivalent if they are developed through completion without any compromising simplifications. The choice of method is made after considering the requirements associated with a particular problem or computer simulation program. This development begins with a Lagrangian approach; then, algebraic manipulation is used to derive the format of equations II-1 through II-5.

Lagrange's equations for the general situation appear as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial D}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j + \sum_{i=1}^{m} a_{ji}\,\lambda_i \qquad \text{(II-18a)}$$

for $(j=1, 2, \cdots n)$

$$\sum_{j=1}^{n} a_{ij}\,\dot{q}_j + a_{it} = 0 \qquad \text{(II-18b)}$$

for $(i=1, 2, \cdots m)$

In these equations, T and V are system kinetic and potential energies, respectively, and D is the Rayleigh dissipation function (accounting for internal damping). The generalized constraint forces

$$\sum_{i} a_{ji}\,\lambda_i$$

augment the generalized forces, Q_j (that arise because of the action of external factors), and are necessary for satisfying the additional conditions of constraint of equation II-18b. The form of equation II-18 is complete and general, in that it includes unconservative forces

(explicitly time dependent), Q_j, and dissipative forces, $\partial D/\partial \dot{q}_i$, and the auxiliary constraint equation (II-18b) are in an all-encompassing form because holonomic conditions may be so represented. The coefficients, (a_{ij}, j = 1, 2, ••, n; t), may depend explicitly on the time, (t); therefore, the constraint conditions as shown account for both rheonomic and scleronomic situations.

In the equations, n is the number of generalized coordinates involved in the representation, and m is the number of auxiliary conditions of constraint. Note that, although the q_j are generalized coordinates (as they must be for the Lagrangian formulation), they are independent *only* in the isolated case when m = 0 or when there are no auxiliary constraint conditions. Some engineers share a misconception on this point: that, if the variables, q_i, are not independent, they are not generalized coordinates. In view of the m constraint equations, this is simply a set of generalized coordinates that are not independent.

In cases where all of the constraint equations are holonomic, it is theoretically possible to eliminate m of the q_i in terms of the remaining n-m. However, if any of the constraint conditions are nonholonomic, a Lagrange multiplier, $\{\lambda_i\}$, must be used in conjunction with that equation. Of course, Lagrange multipliers may be used for either holonomic or nonholonomic constraints.

In that the simulation program includes mathematical representation of active or passive control for elements of the spacecraft system, some state equations involve control variables in addition to equation II-18. The manner in which the additional control equations enter into the composite system state equations is the same whether they are in the form given by equation II-1 or that of equation II-18. In either case, the control system state variables retain their identity, although the control forces/torques necessary to "close the loop" are transformed differently. In the case of Lagrange's equations, the control torques contribute to the generalized forces, Q_j, whereas, in the case of summary equation II-1, they contribute to elements of $\{G\}$ and may be interpreted as ordinary forces or torques, acting at a structural hard point (or at a sensor point). Therefore, discussion of the control system will be postponed until later, and the "mainline" motion equations will be emphasized until the point is reached at which the control system coupling can be clearly indicated.

To "solve" Lagrange's equations of motion, the explicit form of the kinetic and potential energy functions, the dissipation function D, and the form of the transformation relating ordinary Cartesian position coordinates (positioning the typical system particle or element) to the generalized coordinates, q_i must first be defined; the form of the transformation is necessary for expressing generalized forces, Q_j, in terms of external ordinary forces. After the form of the energy functions and coordinate transformation is defined, the indicated differentiations (equation II-18) are performed. A system of ordinary second-order differential equations, which in many cases are nonlinear and which require solution using numerical integration techniques, have been explicitly defined, but the motion equations have not yet been solved.

With numerical implementation and digital programming in mind, the form of the ordinary differential equations is recasted. First of all, they should result in canonical first-order form (the highest time derivatives appear uncoupled on the left-hand side). Also, complicated combinations of generalized velocities and displacements should be grouped so that such groups may be replaced with new variable names. These new variables have been called "quasi-coordinates" in the literature. This will simplify the required computer programming and minimize arithematic computation. Also, it helps considerably in organizing the numerical algorithms necessary for evaluating the left-hand side of the state equations. Thus, recasting the form of the governing equations is sufficiently justified.

The recasting process is begun by defining the forms of kinetic and potential energy and the required transformation. First, note that bodies of the system of flexible bodies are tentatively treated as if they are completely independent of each other. The influence of any body on another is accounted for by the additional constraint conditions and the Lagrange multipliers. Thus, if kinetic and potential energies for the typical body are expressed and Lagrange's equations are applied to it, the ordinary differential equations pertaining to it are simply a subset of equation II-18, and the total system through the representative form of the typical body will have been accounted for.

The generalized coordinates chosen to represent the configuration of the typical body include three Euler angles to indicate attitude of the body fixed-axis system relative to an inertial frame, three projections (components) of the position vector from the origin of the inertial frame to the origin of the body fixed-reference system onto the inertial axes, and N elastic displacement coordinates. Note that the origin of the body fixed-axis system need not necessarily coincide with the body center of mass. Also, the elastic displacement coordinates may be measurements of displacement at a discrete set of points on the body, or they may be coordinates associated with normal vibration modes. In either case, they represent displacements measured in the body axis system. For the r^{th} flexible body, its generalized cooridinates are tabulated as:

$$\{q_r\} = \begin{bmatrix} \phi \\ \theta \\ \psi \\ X \\ Y \\ Z \end{bmatrix} \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Attitude Euler Angles} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Body's Reference Point Position Coordinates} \end{array} \qquad \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \xi_N \end{bmatrix}_r \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{Elastic Displacement Coordinates}$$

A transformation now exists that relates a set of nonholonomic velocities to the generalized velocities that are extensively used in recasting the equations. The transformation appears as:

$$\left[\begin{array}{c} \omega_x \\ \omega_y \\ \omega_z \\ \hline u \\ v \\ w \\ \hline \dot{\xi}_1 \\ \dot{\xi}_2 \\ \cdot \\ \cdot \\ \dot{\xi}_N \end{array}\right] = \left[\begin{array}{ccc|ccc|cccc} \pi_{11} & \pi_{12} & \pi_{13} & & & & & & & \\ \pi_{21} & \pi_{22} & \pi_{23} & & & & & & & \\ \pi_{31} & \pi_{32} & \pi_{33} & & & & & & & \\ \hline & & & \gamma_{11} & \gamma_{12} & \gamma_{13} & & & & \\ & & & \gamma_{21} & \gamma_{22} & \gamma_{23} & & & & \\ & & & \gamma_{31} & \gamma_{32} & \gamma_{33} & & & & \\ \hline & & & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{array}\right] \left[\begin{array}{c} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \hline \dot{X} \\ \dot{Y} \\ \dot{Z} \\ \hline \dot{\xi}_1 \\ \dot{\xi}_2 \\ \cdot \\ \cdot \\ \dot{\xi}_N \end{array}\right] \qquad \text{(II-19)}$$

where, in equation II-19, the vector of nonholonomic velocities, $\{U\}$, contains the three projections, $(\omega_x, \omega_y, \omega_z)$, of the angular velocity vector, $\bar{\omega}$, onto the body fixed axes ($\bar{\omega}$ is the angular velocity of the body reference frame), the three projections of the reference point translational velocity, (u, v, w), onto the body fixed axes, and the displacement rates, $\{\dot{\xi}\}$. The elements of the transformation, γ_{ij} (i, j = 1, 2, 3), are direction cosines; the submatrix, $[\gamma]$, is an orthonormal rotation transformation relating the attitude of the body fixed-axis system to the inertial frame. The submatrix, $[\pi]$, is also a rotation transformation; however, it is not orthonormal because it relates vector components based on an orthogonal basis to those of a skew (nonorthogonal) basis (namely, the axes about which Euler rotations are measured).

In short,

$$\{U\} = [\beta] \; \{\dot{q}\} \qquad \text{(II-20)}$$

Clearly the elements of $[\beta]$ are functions of the three Euler angles. Twelve sets of Euler angles are possible. Any one set is valid for use in subsequent development; the resulting equation form is independent of selection from the twelve sets of angles.

Elements of the transformation, $[\beta]$, may be explicitly defined in terms of three of the generalized coordinates (the Euler angles).

The kinetic-energy expression for the r^{th} body is most easily expressed (initially) in terms of the nonholonomic velocities, $\{U\}$, after which $[\beta]$ is used to replace $\{U\}$ with $[\beta]\ \{\dot{q}\}$. The kinetic energy is then expressed completely in terms of generalized displacements and velocities—the form necessary for applying equation II-18.

Kinetic energy for the typical body is

$$T = \tfrac{1}{2} \int_V \bar{v} \cdot \bar{v} \sigma dV \tag{II-21}$$

where $\bar{v}$ is the velocity field, and σ is mass density, and where integration is carried out over the volume, V, of the body.

The inertial position of any point, p, of the body (figure 3) is:

$$\bar{r} = \bar{X}_R + \bar{\rho}_0 + \bar{\eta} \tag{II-22}$$

where $\bar{X}_R$ is the inertial position of the body's reference point (R is the origin of the body axis system), $\bar{\rho}_0$ positions the point p′ (which coincides with p in the undeformed configuration) from point R, and $\bar{\eta}$ (x, y, z, t) is a measure of elastic displacement.

The vectors $\bar{\rho}_0$ and $\bar{\eta}$ are referenced to the body axis system, thus

$$\bar{\rho}_0 = \lfloor \bar{i} \ \ \bar{j} \ \ \bar{k} \rfloor \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{II-23}$$

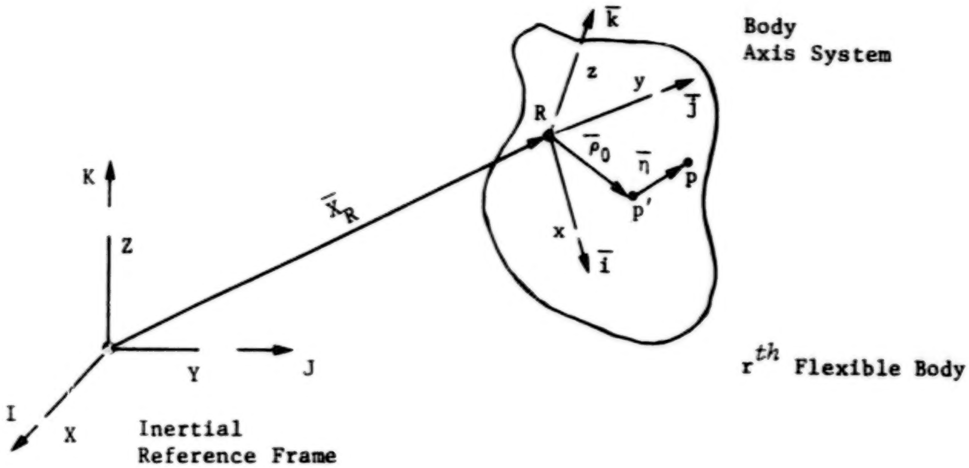


Figure 3. The r^{th} flexible body.

and

$$\bar{\eta}\,(x, y, z, t) = \lfloor \bar{i} \;\; \bar{j} \;\; \bar{k} \rfloor \; \sum_{k=1}^{N} \left(\begin{bmatrix} \phi_{xk}\,(x, y, z) \\ \phi_{yk}\,(x, y, z) \\ \phi_{zk}\,(x, y, z) \end{bmatrix} \xi_k\,(t) \right) \qquad \text{(II-24)}$$

the elastic displacement $\bar{\eta}$ is represented as the superposition of a finite number of single valued space functions, $\bar{\phi}_k$.

The velocity field, $\bar{v}$, is obtained as

$$\bar{v} = \frac{d\bar{r}}{dt} = \bar{v}_R + \bar{\omega} \times \left(\bar{\rho}_o + \bar{\eta} \right) + \sum_{k=1}^{N} \bar{\phi}_k \, \dot{\xi}_k \qquad \text{(II-25)}$$

with

$$\bar{v}_R = \frac{d\bar{X}_R}{dt}$$

The velocity of the reference point, R, may be expressed in terms of components referenced to either the inertial frame or the body frame; that is,

$$\bar{v}_R = \lfloor I \; J \; K \rfloor \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} \tag{II-26}$$

also

$$\bar{v}_R = \lfloor \bar{i} \; \bar{j} \; \bar{k} \rfloor \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

The unit vectors, $\{\bar{i}, \bar{j}, \bar{k}\}$, $\{I, J, K\}$, are related through the rotation transformation, $[\gamma]$, and it follows that

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = [\gamma] \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} \tag{II-27}$$

To be concise, the repeated index summation convention is introduced at this point. With this convention, when any two factors of a term have the same index, summation over the range of that index is implied, and the Σ sign is deleted. For example, the third term on the right of equation II-25 is

$$\bar{\phi}_k \dot{\xi}_k$$

and represents

$$\sum_{k=1}^{N} \bar{\phi}_k \dot{\xi}_k$$

Now, if equation II-25 is substituted into equation II-21, the kinetic energy is

$$
\begin{aligned}
T = \tfrac{1}{2} \int_V \Big\{ & \bar{v}_R \cdot \bar{v}_R + [\bar{\omega} \times (\bar{\rho}_0 + \bar{\eta})] \cdot [\bar{\omega} \times (\bar{\rho}_0 + \bar{\eta})] \\
& + \bar{\phi}_k \cdot \bar{\phi}_j \, \dot{\xi}_k \, \dot{\xi}_j \\
& + 2\, \bar{v}_R \cdot [\bar{\omega} \times (\bar{\rho}_0 + \bar{\eta})] + 2 \bar{v}_R \cdot \bar{\phi}_k \, \dot{\xi}_k \\
& + 2\, [\bar{\omega} \times (\bar{\rho}_0 + \bar{\eta})] \cdot \bar{\phi}_k \, \dot{\xi}_k \Big\} \sigma dV
\end{aligned}
\tag{II-28}
$$

or, integrating term by term over V,

$$
\begin{aligned}
T = \tfrac{1}{2}\, m & \lfloor u \ v \ w \rfloor \{u \ v \ w\} \\
& + \tfrac{1}{2} \lfloor \omega_x \ \omega_y \ \omega_z \rfloor \begin{bmatrix} J_{xx} & -J_{xy} & -J_{xz} \\ -J_{yx} & J_{yy} & -J_{yz} \\ -J_{zx} & -J_{zy} & J_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \\
& + \tfrac{1}{2}\, e_{jk}\, \dot{\xi}_j \, \dot{\xi}_k \\
& + \lfloor u \ v \ w \rfloor \begin{bmatrix} 0 & S_z & -S_y \\ -S_z & 0 & S_x \\ S_y & -S_x & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \\
& + \lfloor u \ v \ w \rfloor \begin{bmatrix} a_{xk} \\ a_{yk} \\ a_{zk} \end{bmatrix} \dot{\xi}_k
\end{aligned}
\tag{II-29}
$$

$$+ \lfloor \omega_x \ \omega_y \ \omega_z \rfloor \begin{bmatrix} d_{xk} \\ d_{yk} \\ d_{zk} \end{bmatrix} \dot{\xi}_k \qquad \text{(II-29) continued}$$

where the following was used:

$$m = \int_V \sigma dV \qquad \text{(II-30)}$$

$$J_{xx} = \int_V [(y + \phi_{yj}\ \xi_j)^2 + (z + \phi_{zj}\ \xi_j)^2]\ \sigma dV \qquad \text{(II-31)}$$

$$= J_{xxo} + 2(b_{yyj} + b_{zzj})\ \xi_j + (c_{yjyk} + c_{zjzk})\ \xi_j\ \xi_k$$

with

$$b_{yyj} = \int_V y\ \phi_{yj}\ \sigma dV$$

$$b_{zzj} = \int_V z\ \phi_{zj}\ \sigma dV \qquad \text{(II-32)}$$

and

$$c_{yjzk} = \int_V \phi_{yj}\ \phi_{zk}\ \sigma dV$$

Also used was

$$a_{xk} = \int_V \phi_{xk} \ \sigma dV \tag{II-33}$$

$$e_{jk} = \int_V \left(\phi_{xj} \ \phi_{xk} + \phi_{yj} \ \phi_{yk} + \phi_{zj} \ \phi_{zk} \right) \ \sigma dV \tag{II-34}$$

and

$$\begin{aligned} S_x &= \int_V \left(x + \phi_{xj} \ \xi_j \right) \sigma dV \\ &= S_{xo} + a_{xj} \ \xi_j \end{aligned} \tag{II-35}$$

$$\begin{aligned} d_{xk} &= \int_V \left[\left(y + \phi_{yj} \ \xi_j \right) \phi_{zk} - \left(z + \phi_{zj} \ \xi_j \right) \phi_{yk} \right] \sigma dV \\ &= b_{yzk} - b_{zyk} + \left(c_{yjzk} - c_{zjyk} \right) \xi_j \end{aligned} \tag{II-36}$$

and also,

$$\begin{aligned} J_{xy} &= \int_V \left(x + \phi_{xj} \ \xi_j \right) \left(y + \phi_{yj} \ \xi_j \right) \sigma dV \\ &= J_{xyo} + \left(b_{xyj} + b_{yxj} \right) \xi_j + c_{xjyk} \ \xi_j \ \xi_k \end{aligned} \tag{II-37}$$

All other quantities involved in equation II-29 are obtained by cyclic permutation of the indexes, x, y, and z. Finally, because the kinetic energy is of quadratic form in the elements of $\{U\}$, it may be expressed as a triple matrix product

$$T = ½ \lfloor U \rfloor [m] \{U\} \tag{II-38}$$

with

$$
[m] = \left[\begin{array}{ccc|ccc|cccc}
J_{xx} & -J_{xy} & -J_{xz} & 0 & -S_z & S_y & d_{x1} & d_{x2} & \cdots & d_{xN} \\
 & J_{yy} & -J_{yz} & S_z & 0 & -S_x & d_{y1} & d_{y2} & \cdots & d_{yN} \\
 & & J_{zz} & -S_y & S_x & 0 & d_{z1} & d_{z2} & \cdots & d_{zN} \\
\hline
 & & & m & 0 & 0 & a_{x1} & a_{x2} & \cdots & a_{xN} \\
 & & & & m & 0 & a_{y1} & a_{x2} & \cdots & a_{yN} \\
 & & & & & m & a_{z1} & a_{z2} & \cdots & a_{zN} \\
\hline
 & \text{(symmetric)} & & & & & e_{11} & e_{12} & \cdots & e_{1N} \\
 & & & & & & & e_{22} & \cdots & e_{2N} \\
 & & & & & & & & \cdots & e_{NN}
\end{array}\right] \tag{II-39}
$$

or in short,

$$
[m] = \left[\begin{array}{c|c|c}
J & -S & d \\
\hline
S & m & a \\
\hline
d^T & a^T & e
\end{array}\right] \tag{II-40}
$$

Using equations II-40, II-19, and II-38 gives

$$
T = \tfrac{1}{2} \lfloor \dot{q} \rfloor [\beta]^T [m] [\beta] \{\dot{q}\} \tag{II-41}
$$

Clearly, the elements of $[m]$ depend only on the ξ_k; the elements of $[\beta]$ depend on the Euler angles, and kinetic energy is therefore a function of generalized velocities and the generalized coordinates themselves. Thus, the functional notation,

$$
T = T\left(q_1, q_2, \cdots q_n ; \dot{q}_1, \dot{q}_2, \cdots \dot{q}_n\right)
$$

is applicable; terms such as $\partial T/\partial q_j$ will come about and play an important role in the simulation.

To continue, it is necessary to express the potential energy, V, and dissipation function, D. Assume that the elastic strain energy can be written as a positive definite quadratic form in the elastic displacement coordinates, or

$$V = \tfrac{1}{2} \lfloor \xi \rfloor [k] \{\xi\} \tag{II-42}$$

the symmetric matrix, [k], is developed by standard finite-element techniques such as those embodied in NASTRAN. If $\{\xi\}$ is a set of normal modal coordinates, then [k] is diagonal with the j^{th} diagonal element appearing as

$$k_{jj} = \omega_j^2 \tag{II-43}$$

where ω_j is the j^{th} natural frequency. Of course, normalization of the eigenvectors (mode shapes) is assumed so that the generalized mass for the j^{th} vibration mode is unity.

Now, because

$$\begin{aligned} \{\xi\} &= [0 \,\vdots\, 0 \,\vdots\, I_N] \{q\} \\ &= [S_\xi] \{q\} \end{aligned} \tag{II-44}$$

it follows that

$$V = \tfrac{1}{2} \lfloor q \rfloor [S_\xi]^T [k] [S_\xi] \{q\} \tag{II-45}$$

Similarly, D is written as

$$D = \tfrac{1}{2} \lfloor \dot{q} \rfloor [S_\xi]^T [C] [S_\xi] \{\dot{q}\} \tag{II-46}$$

where matrix, [C], developed by standard finite-element techniques, is equivalent viscous damping for the structure.

Refer back to Lagrange's equations (II-18a and II-18b), and reexpress them in matrix format as follows:

$$\begin{aligned}\frac{d}{dt}\left([\beta]^T\ [m]\ [\beta]\left\{\dot{q}\right\}\right) = & -[S_\xi]^T\left([k]\ [S_\xi]\left\{q\right\} + [C]\ [S_\xi]\left\{\dot{q}\right\}\right) \\ & + \left\{Q\right\} + \tfrac{1}{2}\left\{\lfloor\dot{q}\rfloor\ [\beta_{,j}]^T\ [m]\ [\beta]\left\{\dot{q}\right\}\right\} \\ & + \tfrac{1}{2}\left\{\lfloor\dot{q}\rfloor\ [\beta]^T\ [m]\ [\beta_{,j}]\left\{\dot{q}\right\}\right\} \\ & + \tfrac{1}{2}\left\{\lfloor\dot{q}\rfloor\ [\beta]^T\ [m_{,j}]\ [\beta]\left\{\dot{q}\right\}\right\} + [a]^T\left\{\lambda\right\}\end{aligned} \tag{II-47}$$

and

$$[a]\left\{\dot{q}\right\} = -\left\{a_t\right\} \tag{II-48}$$

What is meant by $[\beta_{,j}]$ and $[m_{,j}]$ is the partial derivative of each element of $[\beta]$ and $[m]$ with respect to the j^{th} generalized coordinate.

Now, define the ordinary momenta (see Reference Paper II):

$$\begin{aligned}\left\{p\right\} &= [m]\ [\beta]\left\{\dot{q}\right\} \\ &= [m]\left\{U\right\}\end{aligned} \tag{II-49}$$

Also, because

$$\left\{U\right\} = [\beta]\left\{\dot{q}\right\}$$

it follows that

$$\left\{\dot{q}\right\} = [\beta]^{-1}\left\{U\right\} \tag{II-50}$$

Using equations II-49, II-50, II-47, and II-48,

$$\left\{\dot{p}\right\} = -[\beta]^{-1T}[S_\xi]^T\left([k]\ [S_\xi]\left\{q\right\} + [C]\ [S_\xi]\left\{\mathring{q}\right\}\right)$$
$$+ [\beta]^{-1T}\left\{Q\right\} + [\beta]^{-1T}\left(\left\{\lfloor\dot{q}\rfloor\ [\beta_{,j}]^T\left\{p\right\}\right\} - [\mathring{\beta}]^T\left\{p\right\}\right)$$
$$+ \tfrac{1}{2}\ [\beta]^{-1T}\left\{\lfloor U\rfloor\ [m_{,j}]\left\{U\right\}\right\} + [\beta]^{-1T}\ [a]^T\left\{\lambda\right\} \tag{II-51}$$

and

$$[a]\ [\beta]^{-1}\left\{U\right\} = \left\{-a_t\right\} \tag{II-52}$$

On studying equations II-51 and II-52, several observations can be made: First of all, recall the form of $[\beta]$ and $[S_\xi]$ (equations II-19 and II-44). It is clear from these forms that

$$[\beta]^{-1T}\ [S_\xi]^T \equiv [S_\xi]^T \tag{II-53}$$

and that

$$[S_\xi]\left\{q\right\} = \left\{\xi\right\} \tag{II-54}$$

and

$$[S_\xi]\left\{\mathring{q}\right\} = \left\{\mathring{\xi}\right\} \tag{II-55}$$

Also, because the elements of $[m]$ depend only on ξ_k, the first six elements of

$$\left\{\lfloor U\rfloor\ [m_{,j}]\left\{U\right\}\right\}$$

are null; therefore

$$[\beta]^{-1T}\left\{\lfloor U\rfloor\ [m_{,j}]\left\{U\right\}\right\} = \left\{\lfloor U\rfloor\ [m_{,j}]\left\{U\right\}\right\} \tag{II-56}$$

Further, note that matrix $[\beta]^{-1T}$ transforms the generalized forces, $\{Q\}$, to forces "acting in the quasi-coordinates," or call

$$\{G_{ex}\} = [\beta]^{-1T}\{Q\} \tag{II-57}$$

thus, $\{G_{ex}\}$ contains ordinary forces and moments attributable to external sources and corresponds to time derivatives of the ordinary momenta.

Because the transformation, $[\beta]$, depends only on the Euler angles, it follows that only the first six elements of the column

$$[\beta]^{-1T}\left(\left\{\lfloor \dot{q} \rfloor\, [\beta_{,j}]\, \{p\}\right\} - [\dot{\beta}]^{T}\, \{p\}\right)$$

are nonzero, and, after considerable algebraic manipulation, this column may be reexpressed as

$$[\tilde{\Omega}]\, \{p\}$$

or

$$[\tilde{\Omega}]\, \{p\} = \left[\begin{array}{ccc:ccc:c} 0 & \omega_z & -\omega_y & 0 & w & -v & \\ -\omega_z & 0 & \omega_x & -w & 0 & u & \\ \omega_y & -\omega_x & 0 & v & -u & 0 & \\ \hdashline & & & 0 & \omega_z & -\omega_y & \\ & & & -\omega_z & 0 & \omega_x & \\ & & & \omega_y & -\omega_x & 0 & \\ \hdashline & & & & & & \\ & & & & & & \\ & & & & & & \end{array}\right] \left[\begin{array}{c} p(\omega_x) \\ p(\omega_y) \\ p(\omega_z) \\ \hline p(u) \\ p(v) \\ p(w) \\ \hline p(\dot{\xi}_1) \\ \cdot \\ \cdot \\ p(\dot{\xi}_N) \end{array}\right] \tag{II-58}$$

With these observations and definitions, equations II-51 and II-52 may be reexpressed as

$$\{\dot{p}\} = \{G_{ex}\} - \left[\frac{0}{k}\right]\{\xi\} - \left[\frac{0}{C}\right]\{\dot{\xi}\} + [\tilde{\Omega}]\{p\} + \tfrac{1}{2}\left\{\lfloor U \rfloor [m_{,j}] \{U\}\right\} + [b]^T\{\lambda\} \tag{II-59}$$

and

$$[b]\{U\} = \{\dot{\alpha}\} \tag{II-60}$$

where

$$[b] = [a]\,[\beta]^{-1} \tag{II-61}$$

was used, and

$$\{\dot{\alpha}\} = -\{a_t\} \tag{II-62}$$

Note that constraint equation II-60 is now expressed in terms of the nonholonomic velocities, $\{U\}$; the coefficients, [b], are obtained directly from relatively simple, vectorial expressions of kinematic constraint. The same [b] coefficients are transposed and used to multiply $\{\lambda\}$, producing constraint forces/torques corresponding to the ordinary momenta.

If the $\{G\}$ vector is now defined as

$$\{G\} = \{G_{ex}\} - \left[\frac{0}{k}\right]\{\xi\} - \left[\frac{0}{C}\right]\{\dot{\xi}\} + [\tilde{\Omega}]\,[m]\,\{U\} + \tfrac{1}{2}\left\{\lfloor U \rfloor [m_{,j}] \{U\}\right\} - [\dot{m}]\{U\} \tag{II-63}$$

it follows that dynamic equilibrium equations for the typical r^{th} body may be written as

$$\{\dot{U}\}_r = [m]_r^{-1}\left(\{G\}_r + [b]_r^T\{\lambda\}\right) \tag{II-64}$$

to be used in conjunction with system kinematic constraint equations

$$\sum_r [b]_r \{U\}_r = \{\dot{\alpha}\} \tag{II-65}$$

which is the same form as that given by equations II-1 and II-5.

The last three terms of $\{G\}$ given in equation II-63 are inertial forces that involve velocities and displacements of the body. The matrix $[m]$ is an instantaneous inertia matrix, depending on instantaneous values of the deformation coordinates, $\{\xi\}$. The centrifugal and Coriolis effects are completely accounted for within the framework of the assumed velocity field (given by equation II-25). These effects would not be accounted for if "tangential" velocity due to elastic displacement were neglected (i.e., if it were assumed that $|\bar{\omega} \times \bar{\eta}| << |\bar{\omega} \times \bar{\rho}_0|$. In this case, the inertia would be constant and independent of $\{\xi\}$.

An accurate definition of the dynamic equilibrium equations clearly hinges on a complete and accurate definition of the constituents of the $\{G\}_r$ vector, which includes the inertia matrix, $[m]_r$. Also, the kinematic coefficients, $[b]_r$, must be developed in an exact fashion. Kinematics and a more explicit development of $\{G\}$ are given in subsequent sections.

C. Kinematics and System Topology

From a Lagrangian formulation, all of the generalized forces, not derivable from a potential function, ordinarily appear as $\{Q\}$ on the right side of Lagrange's equations of motion. Internal damping forces have been accounted for with the use of Rayleigh's dissipation function, D, and, for generalized constraint forces, by Lagrange's multipliers.

Thus, the remaining generalized forces to deal with include those that are attributable to external factors such as aerodynamic drag, solar pressure, and other commonly encountered environmental loadings.

Control forces (servodrive torques, reaction jets, etc.) are also treated as if they are external. They are not explicitly external, of course, because they depend on time through position and rate errors that are functions of elements of the state vector and on control system state variables that arise from a given control law.

Assume that there is a finite number of points on the typical body at which a force vector (or torque) is known to act. Each of these force/torque vectors contributes to the generalized forces, $\{Q\}$. The generalized forces are calculated by expressing the virtual work of the external ordinary forces in terms of virtual displacements of the points of force application. The transformation that relates ordinary coordinates to generalized coordinates is then used to define the explicit form of the generalized forces. For example, suppose that a force $\bar{f}_p$, and torque, $\bar{T}_p$, act at point p of the typical body. Their virtual work is

$$\delta W = \bar{f}_p \cdot \delta \bar{r}_p + \bar{T}_p \cdot \delta \bar{\theta}_p \tag{II-66}$$

Note that the virtual rotation, $\delta\bar{\theta}_p$, was treated as a vector quantity. This is valid, even though a general rotation is not a vector quantity, because the virtual rotation is infinitesimal and is therefore a vector. Further, because virtual displacements are infinitesimal, $\delta\bar{r}_p$ and $\delta\bar{\theta}_p$ may be expressed in terms of virtual displacements of the quasi-coordinates; that is,

$$\delta\bar{r}_p = \lfloor \bar{i}\ \bar{j}\ \bar{k} \rfloor \left(\begin{bmatrix} \delta r_1 \\ \delta r_2 \\ \delta r_3 \end{bmatrix} + \begin{bmatrix} 0 & (z_p + \eta_{zp}) & -(y_p + \eta_{yp}) \\ -(z_p + \eta_{zp}) & 0 & (x_p + \eta_{xp}) \\ (y_p + \eta_{yp}) & -(x_p + \eta_{xp}) & 0 \end{bmatrix} \begin{bmatrix} \delta\theta_x \\ \delta\theta_y \\ \delta\theta_z \end{bmatrix} + \begin{bmatrix} \phi_{xj}(x_p, y_p, z_p) \\ \phi_{yj}(x_p, y_p, z_p) \\ \phi_{zj}(x_p, y_p, z_p) \end{bmatrix} \delta\xi_j \right) \tag{II-67}$$

and

$$\delta\bar{\theta}_p = \lfloor \bar{i}\ \bar{j}\ \bar{k} \rfloor \left(\begin{bmatrix} \delta\theta_x \\ \delta\theta_y \\ \delta\theta_z \end{bmatrix} + \begin{bmatrix} \sigma_{xj}(x_p, y_p, z_p) \\ \sigma_{yj}(x_p, y_p, z_p) \\ \sigma_{zj}(x_p, y_p, z_p) \end{bmatrix} \delta\xi_j \right) \tag{II-68}$$

where $(\delta r_1, \delta r_2, \delta r_3)$ are components of virtual displacement of the body's reference point, R; $(\delta\theta_x, \delta\theta_y, \delta\theta_z)$ are components of virtual rotation of the body axis system; and $(\sigma_{xj}, \sigma_{yj}, \sigma_{zj})$ are components of the j^{th} space function, $\bar{\sigma}_j$, representing elastic rotation at point, p (modal slopes, for example).

Now, assume that the force and torque vectors, ($\bar{f}_p$ and $\bar{T}_p$), are referenced to the body axis system; they may therefore be written as

$$\bar{f}_p = \lfloor \bar{i}\ \bar{j}\ \bar{k} \rfloor \begin{bmatrix} f_{xp} \\ f_{yp} \\ f_{zp} \end{bmatrix} \tag{II-69}$$

and

$$\overline{T}_p = \lfloor \overline{i} \;\; \overline{j} \;\; \overline{k} \rfloor \begin{bmatrix} T_{xp} \\ T_{yp} \\ T_{zp} \end{bmatrix} \tag{II-70}$$

Note that virtual displacements of the quasi-coordinates are related to virtual generalized displacements by the same transformation that relates nonholonomic velocities to generalized velocities (equation II-19). It follows that the virtual work attributable to $\overline{f}_p$ and $\overline{T}_p$ may be written as

$$\delta W = \lfloor \delta q \rfloor \, [\beta]^T \begin{bmatrix} 1 & & & & -(z_p + \eta_{zp}) & y_p + \eta_{yp} \\ & 1 & & z_p + \eta_{zp} & & -(x_p + \eta_{xp}) \\ & & 1 & -(y_p + \eta_{yp}) & x_p + \eta_{xp} & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ \sigma_{x1})_p & \sigma_{y1})_p & \sigma_{z1})_p & \phi_{x1})_p & \phi_{y1})_p & \phi_{z1})_p \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{xN})_p & \sigma_{yN})_p & \sigma_{zN})_p & \phi_{xN})_p & \phi_{yN})_p & \phi_{zN})_p \end{bmatrix}_{(6 + N \times 6)} \begin{bmatrix} T_{xp} \\ T_{yp} \\ T_{zp} \\ f_{xp} \\ f_{yp} \\ f_{zp} \end{bmatrix} \tag{II-71}$$

The virtual work is also expressed as

$$\delta W = \lfloor \delta q \rfloor \{Q\}$$

and, because δq_j is arbitrary and independent (treated as independent in the face of Lagrange multipliers and constraint equations), it follows that

$$\left\{Q\right\} = [\beta]^T [b_p]^T \begin{Bmatrix} T_p \\ f_p \end{Bmatrix} \tag{II-72}$$

Equations II-71 or II-72 have a noteworthy geometrical interpretation. Note that the first three lines of

$$[b_p]^T \begin{Bmatrix} T_p \\ f_p \end{Bmatrix}$$

are components of the resultant torque vector, $\overline{T}_p + (\overline{\rho}_0 + \overline{\eta}) \times \overline{f}_p$, acting at the body's reference point, R. The second three lines are components of the resultant force vector, $\overline{f}_p$, whereas the j^{th} line, ($j > 6$), corresponds to the standard procedure (of structural dynamicists) for calculating $Q_{\xi j}$, or, as it is usually expressed, generalized forces acting in deformation modes are

$$\left\{Q\right\} = [\phi]^T \left\{T\right\}$$

Also, recalling the form of $[\beta]$ (equation II-19), note that $[\pi]^T$ resolves the resultant torque vector (about orthogonal body axes) to components about skew axes about which Euler rotations are measured, whereas $[\gamma]^T$ resolves the resultant force vector (about orthogonal body axes) to components along the inertial axes. Further, note that $[b_p]$ is a matrix of coefficients that relates the velocity of any point, p, to the vector, $\left\{U\right\}$. This provides additional insight as to why the same coefficients that are used in kinematic constraint equation II-60 are used (in transposed form) to multiply $\left\{\lambda\right\}$-producing resultant constraint forces.

Thus, the remarkable duality of purpose associated with $[b]$-type coefficients has been emphasized. They are initially expressed by writing simple kinematic velocity relationships. The coefficients, $[b]^T$, are then used to transform discrete ordinary forces and torques to equivalent forces and torques acting through the body's reference point, R. The matrix, $[\beta]$, which is also a velocity transformation, is transposed to produce the transformation to generalized forces (if they are desired).

For ordinary momenta equations, the desire is simply to express $\{G_{ex}\}$, which (following equation II-57) is given by

$$\begin{aligned}\{G_{ex}\}_p &= [\beta]^{-1T}\,\{Q\} \\ &= [\beta]^{-1T}\,[\beta]^T\,[b_p]^T \begin{Bmatrix} T_p \\ f_p \end{Bmatrix} \\ &= [b_p]^T \begin{Bmatrix} T_p \\ f_p \end{Bmatrix}\end{aligned} \tag{II-73}$$

This $\{G_{ex}\}_p$ given by equation II-73 reflects only the contribution of the force/torque acting at a single point, p. The total $\{G_{ex}\}$ must be obtained by summing all the points of the body at which forces and torques act, or

$$\{G_{ex}\} = \sum_{i=1}^{NP} [b_{p_i}]^T \begin{Bmatrix} T_{p_i} \\ f_{p_i} \end{Bmatrix} \tag{II-74}$$

Kinematic coefficients, $[b_p]$, such as those of the previous example, will be required throughout in the formulation of the state equations. They are used to synthesize the constraint equations and to produce $\{G\}$, and they are involved in the velocity transformation of equation II-3. It is therefore advantageous to think of a "bank" or collection of all the required kinematic coefficients to be put together in a semiautomatic fashion by using input specifications to the digital program.

1. Sensor Point Kinematics—Force/Torque Transformations

Consider the typical structural hard point, s (figure 4). Assume that a right-handed triad is fixed to point s and that the elements of the triad are unit vectors labeled $\bar{\ell}$, $\bar{m}$, and $\bar{n}$. Now, body, n (which has point s on it), also has a right-handed triad fixed to point n. Suppose that, even when body n is in an undeformed state, the s-triad is misaligned with respect to the n-triad. When the body deforms, there may be further angular misalignment between the two triads. Thus, the relationship linking the two sets of unit vectors is

$$\begin{bmatrix} \bar{\ell} \\ \bar{m} \\ \bar{n} \end{bmatrix} = \left[{}_sR_{s'}\right]\left[{}_{s'}R_n\right] \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} \tag{II-75}$$

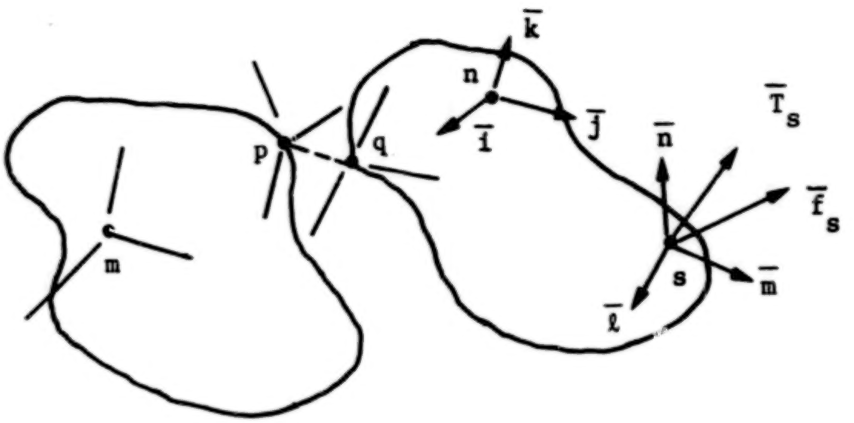


Figure 4. Two typical contiguous bodies of the system.

where $[{}_sR_{s'}]$ and $[{}_{s'}R_n]$ are orthonormal rotation transformations, the first relating the "naturally" misaligned triads via constant Euler rotations and the second accounting for additional rotation due to the body's deformation point s.

The structural deformation at point s is assumed to be sufficiently small that the Euler rotations associated with $[{}_{s'}R_{\hat{n}}]$ may be evaluated through the use of

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = [\sigma_s] \{\xi\} \tag{II-76}$$

where $[\sigma_s]$ is a (3 X N) matrix of modal rotation amplitudes at point s. (Each of the N columns corresponds to a deformation mode.) Concisely denote the traids associated with points n and s by $\{\bar{e}_n\}$ and $\{\bar{e}_s\}$, respectively. The relationship linking the two sets of unit vectors may then be expressed as

$$\{\bar{e}_s\} = \left[{}_sR_n\right] \{\bar{e}_n\} \tag{II-77}$$

In subsequent kinematic development, there is a requirement for expressing the absolute velocity of a typical s-point and the angular velocity of the typical s-triad in terms of velocity states of a given body. Picture a six-long vector (column) of velocity components (three

rotational and three translational) that are projections of $\bar{\omega}_s$ and $\bar{v}_s$ onto the s-traid axes. It is related to the $\{U\}_n$ vector for the body by the transformation

$$\begin{bmatrix} \omega_{xs} \\ \omega_{ys} \\ \omega_{zs} \\ \hline u_s \\ v_s \\ w_s \end{bmatrix}^{(s)} = \left[\begin{array}{c|c|c} \left[{}_sR_n\right] & [0] & \left[{}_sR_n\right]\ [\sigma_s] \\ \hline \left[{}_sR_n\right]\left[S_{ns}^{(n)}\right] & \left[{}_sR_n\right] & \left[{}_sR_n\right]\ [h_s] \end{array}\right] \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ \hline u \\ v \\ w \\ \hline \dot{\xi}_1 \\ \dot{\xi}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{\xi}_N \end{bmatrix}_n^{(n)} \qquad \text{(II-78)}$$

where $[h_s]$ and $[\sigma_s]$ represent matrices of displacement and rotation amplitudes, respectively, and $[S_{ns}^{(n)}]$ is an antisymmetric matrix accounting for a vector cross product, or

$$\left[S_{ns}^{(n)}\right] = \left[\begin{array}{c|c|c} 0 & z_s + \eta_{zs} & -(y_s + \eta_{ys}) \\ -(z_s + \eta_{zs}) & 0 & x_s + \eta_{xs} \\ y_s + \eta_{ys} & -(x_s + \eta_{xs}) & 0 \end{array}\right] \qquad \text{(II-79)}$$

The superscripts in equations II-78 and II-79 are used to indicate the frame to which the velocity components are referenced.

Kinematic coefficients such as those of equation II-78 are generated for each so-called sensor point of the system of bodies. They are used by the simulation program to produce contributions to $\{G_{ex}\}$ from given force/torque components in the manner indicated by equation II-74.

2. *Hinge-Point Kinematics*

Kinematics associated with hinges follows a line of development somewhat similar to that of sensor points. Consider the points, p and q (figure 4), to be two structural hard points associated with a given hinge. All necessary kinematic information pertinent to the hinge is obtained through expressing the velocity of point q relative to point p and in expressing the relative angular velocity between the q- and p-frames. It is convenient that the angular velocity components are projections onto skew axes (Euler angle rates) and that translational velocity components are projections onto the axes of the p-triad. The six relative velocity components may be assembled into a column matrix as

$$\{\dot{\beta}\}_k = \begin{bmatrix} \{\dot{\theta}\} \\ \{\dot{\Delta}\} \end{bmatrix}_k \tag{II-80}$$

where $\{\dot{\theta}\}_k$ represents the three relative Euler angle rates and $\{\dot{\Delta}\}_k$ represents the three relative translational velocity components all of which pertain to the k^{th} hinge. The column of relative velocities may now be expressed as

$$\{\dot{\beta}\}_k = [b_p]_k \{U\}_m + [b_q]_k \{U\}_n \tag{II-81}$$

with

$$[b_p] = \left[\begin{array}{c|c|c} -[\pi]^{-1}\,[{}_qR_p]\,[{}_pR_m] & [0] & -[\pi]^{-1}\,[{}_qR_p]\,[{}_pR_m]\,[\sigma_p] \\ \hline -[{}_pR_m]\,[S^{(m)}_{mp}] & -[{}_pR_m] & -[{}_pR_m]\,[h_p] \end{array}\right] \tag{II-82}$$

and

$$[b_q] = \left[\begin{array}{c|c|c} [\pi]^{-1}\,[{}_qR_n] & [0] & [\pi]^{-1}\,[{}_qR_n]\,[\sigma_q] \\ \hline [{}_pR_q]\,[{}_qR_n]\,[S^{(n)}_{nq}] & [{}_pR_q]\,[{}_qR_n] & [{}_pR_q]\,[{}_qR_n]\,[h_q] \end{array}\right] \tag{II-83}$$

In equations II-82 and II-83, the rotation transformations, $[{}_pR_m]$ and $[{}_qR_n]$, are developed to include the effects of structural deformation as indicated in equation II-75; the rotation

transformations, $[\pi]^{-1}$ and $[{}_pR_q]$, are developed in standard fashion using the three Euler rotations, $\{\theta\}_k$.

For further discussion, consider the system of bodies shown in figure 5. Topology of the system is simply indicated by an integer array, called "ITOPOL," as follows:

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \longleftarrow \text{Hinge number} \\
[\text{ITOPOL}] = & \begin{array}{|cccccccc|} \hline 1 & 2 & 4 & 3 & 5 & 6 & 7 & 7 \\ \hline 0 & 3 & 2 & 6 & 3 & 1 & 5 & 2 \\ \hline \end{array} \begin{array}{l} \longleftarrow \text{Body (n) relative to} \\ \longleftarrow \text{Body (m)} \end{array}
\end{array}
$$

The [ITOPOL] array, which is the actual input to the simulation program, is used to define system topology as indicated. Now, with reference to the example shown in figure 5 and the corresponding (ITOPOL) array, the form of the velocity transformation may be written

$$
\begin{array}{c|c}
\text{Hinge} \backslash \text{Body} & \begin{array}{ccccccc} (1) & (2) & (3) & (4) & (5) & (6) & (7) \end{array} \\
\begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \\ (7) \\ (8) \end{array} &
\left[\begin{array}{c|c|c|c|c|c|c}
b_{q_{1,1}} & & & & & & \\ \hline
 & b_{q_{2,2}} & b_{p_{2,3}} & & & & \\ \hline
 & b_{p_{3,2}} & & b_{q_{3,4}} & & & \\ \hline
 & & b_{q_{4,3}} & & & b_{p_{4,6}} & \\ \hline
 & & b_{p_{5,3}} & & b_{q_{5,5}} & & \\ \hline
b_{p_{6,1}} & & & & & b_{q_{6,6}} & \\ \hline
 & & & & b_{p_{7,5}} & & b_{q_{7,7}} \\ \hline
 & b_{p_{8,2}} & & & & & b_{q_{8,7}}
\end{array}\right]
\end{array}
\begin{bmatrix} \{U\}_1 \\ \{U\}_2 \\ \{U\}_3 \\ \{U\}_4 \\ \{U\}_5 \\ \{U\}_6 \\ \{U\}_7 \end{bmatrix}
=
\begin{bmatrix} \{\dot\beta\}_1 \\ \{\dot\beta\}_2 \\ \{\dot\beta\}_3 \\ \{\dot\beta\}_4 \\ \{\dot\beta\}_5 \\ \{\dot\beta\}_6 \\ \{\dot\beta\}_7 \\ \{\dot\beta\}_8 \end{bmatrix}
\qquad \text{(II-84)}
$$

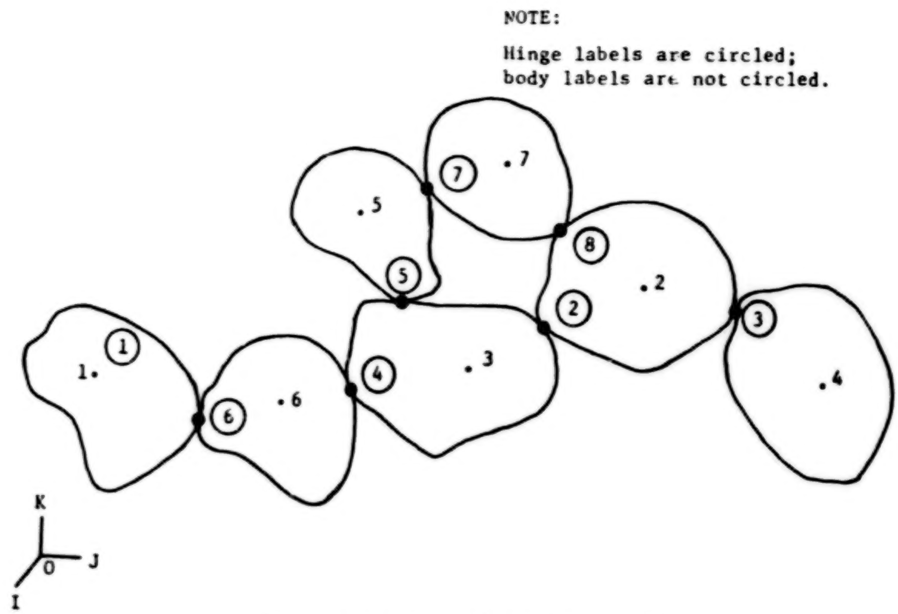


Figure 5. Topology of a typical system.

where $[b_{p_{i,j}}]$ and $[b_{q_{i,j}}]$ are matrices as defined in equations II-82 and II-83 (with i = Hinge number and j = Body number). The velocity transformations of equation II-84 represent the "bank" of all hinge kinematic coefficients previously mentioned and produces every possible velocity component pertinent to hinges. Referring to basic system equations II-3 and II-5, note that selected lines, or equations, from the bank (equation II-84) are taken to represent constraint equations or position coordinate rate equations. The $[B]_j$ and $[b]_j$ coefficients of equations II-3 and II-5 are simply subpartitions extracted from equation II-84.

To implement calculation of Lagrange's multipliers (equation II-6), it is necessary to develop time derivatives of $[b]_j$ coefficients. In a manner similar to the foregoing, in which all $[b]_j$ coefficients are extracted from the complete collections, the $[\dot{b}]_j$ matrices come from a collection of matrices whose members are $[\dot{b}_{q_{i,j}}]$ and $[\dot{b}_{p_{i,j}}]$, which are developed in Appendix C.

D. Development of the $\{G\}_j$ Force Vector

The equations of dynamic equilibrium for the j^{th} body of the system are given in an earlier section as equation II-1. As noted there, the right-hand side includes a so-called $\{G\}_j$ vector, which accounts for all state-dependent forces except for those of interconnection constaint. In equation II-63, the $\{G\}_j$ vector is presented in a somewhat more-developed form.

The purpose of this section is to provide more explicit development of the elements that contribute to $\{G\}_j$. All contributions in the following expression may be accounted for (omitting the j subscript, understanding that the typical, or j^{th} body, is being dealt with):

$$\{G\} = \{G_{ex}\} - \begin{bmatrix} 0 \\ \hline k \end{bmatrix} \{\xi\} - \begin{bmatrix} 0 \\ \hline C \end{bmatrix} \{\dot{\xi}\} + [\tilde{\Omega}]\,[m]\,\{U\}$$
$$+ \tfrac{1}{2} \left\{ \lfloor U \rfloor\,[m_{,k}]\,\{U\} \right\} - [\dot{m}]\,\{U\} + \{G_{mw}\} + \{G_{gg}\} \qquad \text{(II-85)}$$

Although the first term, $\{G_{ex}\}$, has been discussed in the previous section (equation II-74), note here that the ordinary force/torque components that produce $\{G_{ex}\}$ may be considered as a miscellaneous force vector. Its presence provides the program user latitude for including a variety of additional effects. Clearly, it is the implement through which control forces/torques are "fed back" to the dynamic system.

The second and third terms of equation II-85 have been previously introduced. There is no implicit restriction on the stiffness and damping matrices, [k] and [C], nor is there a restriction on definition of the $\{\xi\}$ coordinates; in the majority of cases, they will likely be coordinates associated with orthonormal vibration modes. However, they may be physical (ordinary-discrete) displacement coordinates as well. In the latter case, the [k] and [C] matrices are usually coupled.

The last two terms of equation II-85 are included to account for momentum-wheel coupling and gravity effects, respectively. The treatment given to built-in momentum wheels is such that, in addition to producing a contribution to $\{G\}$, there is also a required extension to the form of the $[m]_j$ matrices because momentum wheels are *inertially* coupled. Thus, there is sufficient requirement for a dedicated development concerning momentum wheels. The next two sections deal exclusively with momentum-wheel and gravity effects, respectively.

The remaining terms that contribute to $\{G\}$ are basic inertial effects and involve the matrices, $[m]$, $[m_{,k}]$, and $[\dot{m}]$. With reference to equation II-39, the form, [m], is given, corresponding to the case with available single-valued space functions $\bar{\phi}_k$. Ordinarily, access to such a description of the structure's deformation modes is not possible because of the structural complexity of typical spacecraft. The analyst should always be able to obtain, as data, matrices of modal amplitude ratios (mode shapes) and the corresponding structural mass matrix (generated by finite element techniques). To accommodate data based on the more practical definition of structural characteristics, it is necessary to recast the inertia matrices, [m], in a similar but more general format. The generality of the development of paragraph II.B is not compromised by extending the form of the inertia matrix. The extended, or more general, inertia matrix is developed in Appendix A, but here, for purposes of developing

inertial contributions to the $\{G\}$ vector, the resulting form is accepted and the kinetic energy expression is presented as

$$T = \tfrac{1}{2} \lfloor U \rfloor \left([m_o] + [m_1]_j \xi_j + [m_2]_{jk} \xi_j \xi_k \right) \{U\} \tag{II-86}$$

with the repeated index summation convention implied, and with $[m_o]$ of the form

$$[m_o] = \left[\begin{array}{c|c|c} J & -S & d \\ \hline S^T & m & a \\ \hline d^T & a^T & e \end{array}\right] \tag{II-87}$$

that is, it is identical to the $[m]$ given by equation II-39 except that it is constant, independent of deformation. The constant inertia matrix, $[m_o]$, as given by equation II-87, is always of the form shown regardless of the choice of "modal" columns. The form of the matrices, $[m_1]$ and $[m_2]$, is such as to accommodate the general situation; that is, their definition includes inertial integrals as defined for a continuous system (equations II-30 through II-37) or as defined by structural mass matrices that are called "lumped" or "consistent."

The inertia matrix associated with ξ_j is

$$[m_1]_j = \left[\begin{array}{ccc|ccc|c} 2b_1 & -b_4 & -b_5 & \alpha_1 & \alpha_2 & \alpha_3 & \lfloor (C_{yz})_{jk} \rfloor \\ & 2b_2 & -b_6 & \alpha_4 & \alpha_5 & \alpha_6 & \lfloor (C_{zx})_{jk} \rfloor \\ & & 2b_3 & \alpha_7 & \alpha_8 & \alpha_9 & \lfloor (C_{xy})_{jk} \rfloor \\ \hline & & & 0 & 0 & 0 & 0\ 0\ 0 \\ & & & & 0 & 0 & 0\ 0\ 0 \\ & & & & & 0 & 0\ 0\ 0 \\ \hline & & & & & & 0\ 0\ 0 \\ & \text{(symmetric)} & & & & & 0\ 0 \\ & & & & & & 0 \end{array}\right]_j \tag{II-88}$$

and the one associated with $\xi_j\, \xi_k$ is

$$[m_2]_{jk} = \left[\begin{array}{ccc|c|c} C_{11} & -C_{12} & -C_{13} & & \\ & C_{22} & -C_{23} & 0 & 0 \\ & & C_{33} & & \\ \hline & & & 0 & 0 \\ \hline \multicolumn{3}{l|}{\text{(symmetric)}} & & 0 \end{array}\right]_{jk} \qquad \text{(II-89)}$$

Now, for N-deformation modes associated with a given body, it is understood that the range of the indices, j and k, is N; the coefficients, $(C_{11})_{jk}$, $(C_{12})_{jk}$, $\cdots$ $(C_{xy})_{jk}$, are therefore stored as 9 ($N \times N$) arrays of inertial integrals, whereas $(b_1)_j$, $(b_2)_j$, $\cdots$ $(b_6)_j$ and $(\alpha_1)_j$, $(\alpha_2)_j$, $\cdots$ $(\alpha_9)_j$ are stored as a ($6 \times N$) array and a ($9 \times N$) array, respectively. Thus, from a programming standpoint, note that $9N^2 + 15N$ storage locations are required for accommodating the inertial integrals necessary to account for the deformation-dependent mass matrix. Of course, if a particular body is rigid ($N = 0$), then only the first (6×6) diagonal partition of $[m_o]$ is used.

When the body is flexible ($N > 0$), the inertia matrix is calculated from deformation states (ξ_j) and inertia integrals in the manner indicated by equation II-86; the redundant operations due to symmetry and null operations are avoided in the digital code.

Having an instantaneous numerical evaluation of the inertia matrix, the term, $[\tilde{\Omega}]\,[m]\,\{U\}$, is calculated and added to $\{G\}$, consistent with the expression of equation II-58.

It is now possible to express explicitly, the combination of the remaining two inertial force vectors in terms of the inertial integrals given in equations II-88 and II-89. For further development, the combination may be defined as

$$\{G_c\} = \left\{ \lfloor U \rfloor \; [m_{,k}] \; \{U\} \right\} - [\dot{m}] \; \{U\} \qquad \text{(II-90)}$$

Thus, the first element of $\{G_c\}$, corresponding to ω_x, is

$$(G_c)_1 = \Big\{ -2\omega_x (b_1)_j + \omega_y (b_4)_j + \omega_z (b_5)_j$$
$$-u(\alpha_1)_j - v(\alpha_2)_j - w(\alpha_3)_j \tag{II-91}$$
$$-(C_{yz})_{jk} \dot{\xi}_k - 2\omega_x (C_{11})_{\ell j} \xi_\ell$$
$$+\omega_y [(C_{12})_{\ell j} + (C_{12})_{j\ell}] \xi_\ell + \omega_z [(C_{13})_{\ell j} + (C_{13})_{j\ell}] \xi_\ell \Big\} \dot{\xi}_j$$

The second element, corresponding to ω_y, is

$$(G_c)_2 = \Big\{ \omega_x (b_4)_j - 2\omega_y (b_2)_j + \omega_z (b_6)_j$$
$$-u(\alpha_4)_j - v(\alpha_5)_j - w(\alpha_6)_j \tag{II-92}$$
$$-(C_{zx})_{jk} \dot{\xi}_k + \omega_x [(C_{12})_{\ell j} + (C_{12})_{j\ell}] \xi_\ell$$
$$-2\omega_y (C_{22})_{\ell j} \xi_\ell + \omega_z [(C_{23})_{\ell j} + (C_{23})_{j\ell}] \xi_\ell \Big\} \dot{\xi}_j$$

The third element, corresponding to ω_z, is

$$(G_c)_3 = \Big\{ \omega_x (b_5)_j + \omega_y (b_6)_j - 2\omega_z (b_3)_j$$
$$-u(\alpha_7)_j - v(\alpha_8)_j - w(\alpha_9)_j \tag{II-93}$$
$$-(C_{xy})_{jk} \dot{\xi}_k + \omega_x [(C_{13})_{\ell j} + (C_{13})_{j\ell}] \xi_\ell$$
$$+ \omega_y [(C_{23})_{\ell j} + (C_{23})_{j\ell}] \xi_\ell - 2\omega_z (C_{33})_{\ell j} \xi_\ell \Big\} \dot{\xi}_j$$

The fourth element, corresponding to u, is

$$(G_c)_4 = -\left\{\omega_x (\alpha_1)_j + \omega_y (\alpha_4)_j + \omega_z (\alpha_7)_j\right\} \dot{\xi}_j \tag{II-94}$$

The fifth element, corresponding to v, is

$$(G_c)_5 = -\left\{\omega_x (\alpha_2)_j + \omega_y (\alpha_5)_j + \omega_z (\alpha_8)_j\right\} \dot{\xi}_j \tag{II-95}$$

The sixth element, corresponding to w, is

$$(G_c)_6 = -\left\{\omega_x (\alpha_3)_j + \omega_y (\alpha_6)_j + \omega_z (\alpha_9)_j\right\} \dot{\xi}_j \tag{II-96}$$

Finally, for the element k + 6, corresponding to an inertial force acting in the ξ_k coordinate,

$$\begin{aligned}
(G_c)_{k+6} &= \omega_x^2 \, [(C_{11})_{kj} \, \xi_j + (b_1)_k] \\
&+ \omega_y^2 \, [(C_{22})_{kj} \, \xi_j + (b_2)_k] \\
&+ \omega_z^2 \, [(C_{33})_{kj} \, \xi_j + (b_3)_k] \\
&- \omega_x \omega_y \left\{ [(C_{12})_{kj} + (C_{12})_{jk}] \, \xi_j + (b_4)_k \right\} \\
&- \omega_x \omega_z \left\{ [(C_{13})_{kj} + (C_{13})_{jk}] \, \xi_j + (b_5)_k \right\} \\
&- \omega_y \omega_z \left\{ [(C_{23})_{kj} + (C_{23})_{jk}] \, \xi_j + (b_6)_k \right\} \\
&+ \omega_x \, [(\alpha_1)_k \, u + (\alpha_2)_k \, v + (\alpha_3)_k \, w]
\end{aligned} \tag{II-97}$$

$$+ \omega_y \left[(\alpha_4)_k u + (\alpha_5)_k v + (\alpha_6)_k w\right]$$

$$+ \omega_z \left[(\alpha_7)_k u + (\alpha_8)_k v + (\alpha_9)_k w\right]$$

$$+ \left\{ \omega_x \left[(C_{yz})_{kj} - (C_{yz})_{jk}\right] + \omega_y \left[(C_{zx})_{kj} - (C_{zx})_{jk}\right] \right. \qquad \text{(II-97) continued}$$

$$\left. + \omega_z \left[(C_{xy})_{kj} - (C_{xy})_{jk}\right] \right\} \dot{\xi}_j$$

On examining the composition of the inertial force $(G_c)_{k+6}$, note that the first six bracketed terms represent centrifugal forces (distance × omega-squared) acting in the deformation coordinates, whereas the last bracketed terms of equation II-97 represent Coriolis forces (velocity × omega).

E. Momentum-Wheel Coupling

The spacecraft system undergoing analysis may have several "built-in" momentum wheels. A momentum wheel is usually defined as a cylindrical or disk-shaped mass that spins about an axis that is fixed to a structural hard point of a given body. The wheel can be either spun up or despun by an electric motor whose rotor is part of the rotating mass. The shaft torque that acts to accelerate the wheel also acts on the body in a negative sense, providing active attitude control. The shaft torque is generally governed by a control law that "senses" attitude and rate errors of the body. In this development, a momentum wheel is assumed to be inertially symmetric about its spin axis.

To develop the inertial coupling effects of the typical momentum wheel, consider three unit-vector bases:

$$\lfloor \bar{e}_n \rfloor = \lfloor \bar{i}, \bar{j}, \bar{k} \rfloor \qquad \text{(II-98)}$$

$$\lfloor \bar{e}_s \rfloor = \lfloor \bar{\ell}, \bar{m}, \bar{n} \rfloor \qquad \text{(II-99)}$$

$$\lfloor \bar{e}_w \rfloor = \lfloor \bar{\ell}', \bar{m}', \bar{n}' \rfloor \qquad \text{(II-100)}$$

The first triad is the body-reference triad for body n, the second is a sensor-point triad (fixed to point s), and the third triad is fixed in the momentum wheel. One of the three unit vectors of $\lfloor \bar{e}_s \rfloor$ is coincident with one of the unit vectors of $\lfloor \bar{e}_w \rfloor$ (i.e., either $\bar{\ell}$, $\bar{m}$, or $\bar{n}$ may be the spin axis depending on the preference of the analyst). In figure 6, $\bar{n} = \bar{n}'$ is shown as the common, or spin, axis.

The absolute angular velocity of the $\lfloor \bar{e}_w \rfloor$ frame can be expressed as

$$\bar{\omega}_w = \lfloor \bar{e}_w \rfloor \; [{}_wR_s] \left(\{\omega_s\} + \{P_w\} \, \dot{\theta} \right) \tag{II-101}$$

where $\{P_w\}$ is an elementary three-long position vector (null except for unity in the first, second, or third locations corresponding to $\bar{\ell}$, $\bar{m}$, or $\bar{n}$ being the spin axis), and θ is the relative angular speed of the $\lfloor \bar{e}_w \rfloor$ frame with respect to the $\lfloor \bar{e}_s \rfloor$ frame.

With the inertial characteristics assumed (axisymmetry) for the wheel and with the velocity expression of equation II-101, the total angular momentum vector for the wheel may be written as

$$\begin{aligned} \bar{h} &= \lfloor \bar{e}_w \rfloor \; [J_w] \; \{\omega_w\} \\ &= \lfloor \bar{e}_s \rfloor \; [J_w] \left(\{\omega_s\} + \{P_w\} \, \dot{\theta} \right) \end{aligned} \tag{II-102}$$

with $[J_w]$ diagonal with all diagonal values equal to J_T except for the position corresponding to the spin axis, J_s. J_T is the mass moment of inertia about any axis perpendicular to the spin axis, and J_s is the spin inertia for the wheel.

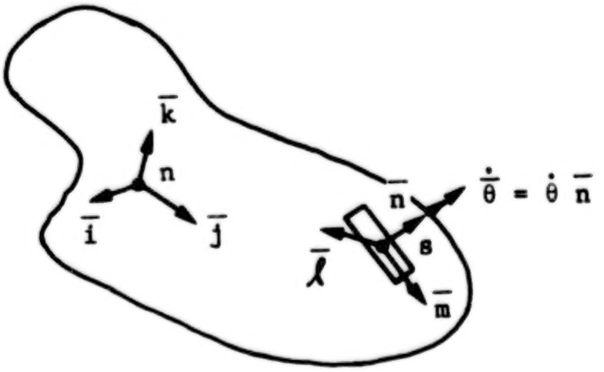


Figure 6. Typical body/momentum-wheel relationship.

The torque acting on the wheel (resolved to the $[\bar{e}_s]$ frame) is

$$\bar{T} = \lfloor e_s \rfloor \{T\} = \frac{d}{dt}\bar{h}$$
$$= \lfloor e_s \rfloor \left([J_w] \{\dot{\omega}_s\} + \{P_w\} J_s \ddot{\theta} - [\Omega_s] [J_w] \{\omega_s\} - [\Omega_s] \{P_w\} J_s \dot{\theta} \right) \tag{II-103}$$

where an SK* operator is defined so that

$$[\Omega_s] = SK^* \{\omega_s\}$$

or

$$\begin{bmatrix} 0 & \omega_{s3} & -\omega_{s2} \\ -\omega_{s3} & 0 & \omega_{s1} \\ \omega_{s2} & -\omega_{s1} & 0 \end{bmatrix} = SK^* \begin{bmatrix} \omega_{s1} \\ \omega_{s2} \\ \omega_{s3} \end{bmatrix} \tag{II-104}$$

The torque acting on body n at point s due to the wheel is $-\bar{T}$, and it drives the body's quasi-coordinate as

$$\{G'_{mw}\} = - [\hat{b}_s]^T \left([J_w] [\hat{b}_s] \{\dot{U}\}_n + [J_w] [\dot{\hat{b}}_s] \{U\}_n + \{P_w\} J_s \ddot{\theta} - [\Omega_s] [J_w] \{\omega_s\} - [\Omega_s] \{P_w\} J_s \dot{\theta} \right) \tag{II-105}$$

with

$$[\hat{b}_s] = \left[{}_sR_n\right] \left[I \,|\, 0 \,|\, \sigma_s \right] \tag{II-106}$$

and also, as can easily be shown,

$$[\dot{\hat{b}}_s] \{U\}_n = [SK^* (\left[{}_sR_n\right] [\sigma_s] \{\dot{\xi}\}_n)] [\hat{b}_s] \{U\}_n \tag{II-107}$$

Now, the shaft torque is simply the projection of $\bar{T}$ onto the spin axis, or

$$
\begin{aligned}
T_s &= \lfloor P_w \rfloor \{T\} \\
&= J_s \lfloor P_w \rfloor [\hat{b}_s] \{\dot{U}\}_n + J_s \lfloor P_w \rfloor [\dot{\hat{b}}_s] \{U\}_n + J_s \ddot{\theta}
\end{aligned}
\qquad \text{(II-108)}
$$

Equations II-105 and II-108 permit the coupled equations for body n and several momentum wheels to be expressed as

$$
\left[\begin{array}{c|c|c}
m_n + \hat{b}_1^T J_{w1} \hat{b}_1 + \hat{b}_2^T J_{w2} \hat{b}_2 & \hat{b}_1^T P_{w1} J_{s1} & \hat{b}_2^T P_{w2} J_{s2} \\
\hline
J_{s1} P_{w1}^T \hat{b}_1 & J_{s1} & 0 \\
\hline
J_{s2} P_{w2}^T \hat{b}_2 & 0 & J_{s2}
\end{array}\right]
\left[\begin{array}{c}
\{\dot{U}\}_n \\ \hline \ddot{\theta}_1 \\ \hline \ddot{\theta}_2
\end{array}\right]
=
$$

$$
\left[\begin{array}{c}
\{\hat{G}\}_n + [b]_n^T \{\lambda\} \\ \hline 0 \\ \hline 0
\end{array}\right]
+
\left[\begin{array}{c}
\sum_w [\hat{b}_s]^T [\Omega_s] [J_w] \{\omega_s\} - \sum_w [\hat{b}_s]^T [J_w] [\dot{\hat{b}}_s] \{U\}_n \\ \hline 0 \\ \hline 0
\end{array}\right]
\qquad \text{(II-109)}
$$

$$
+
\left[\begin{array}{c}
-\sum_w [b_s]^T (SK^{*} \{P_w\}) [\hat{b}_s] \{U\}_n J_s \dot{\theta} \\ \hline
T_{s1} - J_{s1} \lfloor P_{w1} \rfloor [\dot{\hat{b}}_1] \{U\}_n \\ \hline
T_{s2} - J_{s2} \lfloor P_{w2} \rfloor [\dot{\hat{b}}_2] \{U\}_n
\end{array}\right]
$$

The inertially coupled body/momentum-wheel equations (for two wheels) are shown as equation II-109 simply for the purpose of indicating the form. Note that, within the equations, there effectively resides the original form of the dynamic-equilibrium equations for body n; namely,

$$
[m]_n \{\dot{U}\}_n = \{\hat{G}\}_n + [b]_n^T \{\lambda\} \qquad \text{(II-110)}
$$

which govern if no momentum wheels are associated with body n. In equation II-110, the caret ($\wedge$) has been placed over G to represent the right-hand side force vector that excludes momentum-wheel effects.

Now, on further study of the form of equation II-109, note that, if the "locked" momentum-wheel effects are already included in the definition of $[m]_n$ (which is the standard practice when inertially coupling systems together), the (1, 1) partition of the coefficients on the left of equation II-109 becomes simply $[m]_n$. Also, the second column on the right of equation II-109 is absorbed in $\{\hat{G}\}_n$, having already been accounted for in the development of dynamic-equilibrium equations.

It therefore follows that, in order to implement momentum-wheel coupling with one of the flexible bodies, it is only necessary to extend the $\{U\}_n$ vector to contain momentum-wheel spin values, ($\dot{\theta}$), to extend the inertia (except for the [1, 1] partition) as indicated in equation II-109 and to add to the right-hand side force vector

$$\{G_{mw}\} = - \left[\begin{array}{c} \sum_{w} [\hat{b}_s]^T (SK^{\bullet} \{P_w\}) [\hat{b}_s] \ \{U\}_n J_s \dot{\theta} \\ \hdashline T_{s1} - J_{s1} \lfloor P_{w1} \rfloor [\dot{\hat{b}}_1] \{U_n\} \\ \hdashline T_{s1} - J_{s2} \lfloor P_{w2} \rfloor [\dot{\hat{b}}_2] \{U_n\} \end{array} \right] \tag{II-111}$$

The values for shaft torque T_s that appear in $\{G_{mw}\}$ are established by a given control law if the wheels are to be considered variable speed. If a given momentum wheel is of constant speed (used only for "gyroscopic damping"), the torque equation for it is deleted from the form of equation II-109; however, its effects are still included in the upper partition of the vector, $\{G_{mw}\}$ (the gyroscopic torque due to constant $\dot{\theta}$).

Clearly, the equations of dynamic equilibrium for a body, after having been augmented to include momentum-wheel coupling, are still of the general form

$$\{\dot{U}\}_j = [m]_j^{-1} \left(\{G\}_j + [b]_j^T \{\lambda\} \right) \tag{II-112}$$

F. Gravity-Gradient Effects

Attitude dynamics of orbiting spacecraft can be significantly influenced by the gravitational force that is distributed according to the system's position and deformation state. The gravitational force per unit mass varies (in a central force field) simply because different mass particles are at different distances from the Earth's center of mass. Figure 7 describes the geometry associated with a typical elastic body.

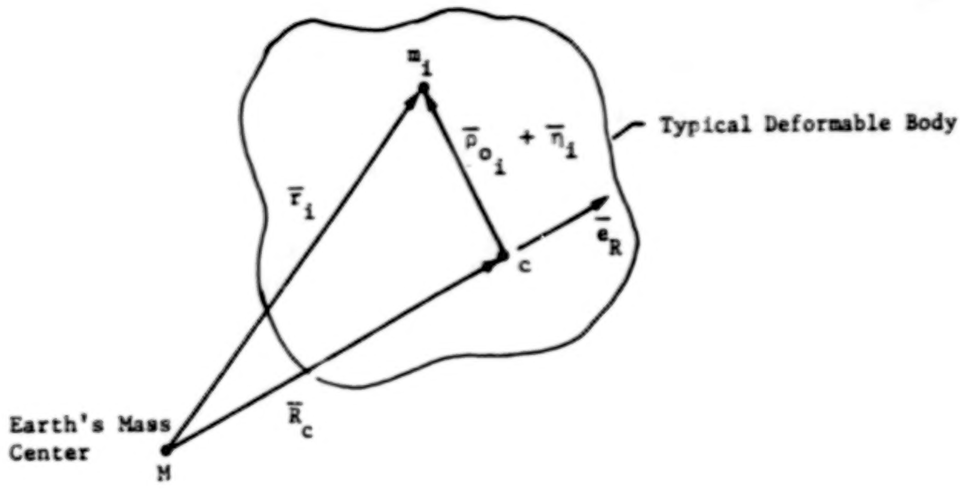


Figure 7. Geometry for gravity effects on a typical body.

For a central force field, the gravitational force per unit mass is given as

$$\left(\frac{\bar{F}}{m}\right)_i = -\frac{GM}{r_i^2}\frac{\bar{r}_i}{r_i} \tag{II-113}$$

which, to a first-order approximation, is

$$\left(\frac{\bar{F}}{m}\right)_i = -g_c\left[\bar{e}_R + \frac{\bar{\rho}_{o_i} + \bar{\eta}_i}{R_c} - 3\,\bar{e}_R\left(\bar{e}_R \cdot \frac{\bar{\rho}_{o_i} + \bar{\eta}_i}{R_c}\right)\right] \tag{II-114}$$

where

GM = the Earth's gravitational constant

m_i = the typical mass particle

g_c = local gravitational acceleration

$\bar{e}_R$ = a unit vector directed along $\bar{R}_c$

c = the origin of the body reference system

The virtual work due to gravitational force can be written as

$$\delta W_g = \sum_i \left(\frac{\bar{F}}{m}\right)_i \cdot \delta \bar{r}_i \, m_i$$

$$= \int_V \left(\frac{\bar{F}}{m}\right) \cdot \delta \bar{r} \, \sigma dV \tag{II-115}$$

with m_i replaced by differential mass σdV.

The virtual displacement field is expressed in terms of virtual displacements of the quasi-coordinates as

$$\delta \bar{r} = \delta \bar{r}_c + \delta \bar{\theta}_c \times (\bar{\rho}_0 + \bar{\eta}) + \delta \bar{\eta} \tag{II-116}$$

In combining equation II-115 with equation II-116, the torque about point c, due to gravity-gradient effects, is

$$\left(\bar{T}_c\right)_g = g_c \bar{e}_R \times \bar{S} + \frac{3g_c}{R_c} \bar{e}_R \times \left(\bar{\bar{J}} \cdot \bar{e}_R\right) \tag{II-117}$$

where

$\bar{S}$ = the first mass moment about point c

$\bar{\bar{J}}$ = the instantaneous inertia tensor (deformation dependent) for the body

The resultant force due to gravity effects is

$$\left(\bar{F}_c\right)_g = -g_c m \bar{e}_R - \frac{g_c}{R_c} \bar{S} + \frac{3g_c}{R_c} \left(\bar{e}_R \cdot \bar{S}\right) \bar{e}_R \tag{II-118}$$

and the force acting in the k^{th} deformation coordinate, ξ_k, is

$$\left(G_{\xi_k}\right)_g = -g_c \left\{ \int_V \bar{\phi}_k \cdot \bar{e}_R \sigma dV + \frac{1}{R_c} \int_V \bar{\phi}_k \cdot \left(\bar{\rho}_o + \bar{\eta}\right) \sigma dV \right.$$

$$\left. -3 \frac{\bar{e}_R}{R_c} \cdot \int_V \bar{\phi}_k \left[\bar{e}_R \cdot \left(\bar{\rho}_o + \bar{\eta}\right)\right] \sigma dV \right\} \tag{II-119}$$

Now, the unit vector, $\bar{e}_R$, has projections onto the body axis system that continually vary as the body changes attitude. The unit vector, $\bar{e}_R$, is expressed in terms of direction cosines, and the three unit vectors associated with the body-reference frame are expressed as

$$\bar{e}_R = \left\lfloor \bar{e}_B \right\rfloor \left\{\gamma_g\right\} \tag{II-120}$$

Also defined are

$$\left[\tilde{\gamma}_g\right] = SK^* \left\{\gamma_g\right\} \tag{II-121}$$

$$\bar{S} = \left[\bar{e}_B\right] \left\{S\right\} \tag{II-122}$$

$$\left[\tilde{S}\right] = SK^* \left\{S\right\} \tag{II-123}$$

$$\left\{a\right\}_k = \int_V \left\{\phi\right\}_k \sigma dV \tag{II-124}$$

With these definitions and the force and torque expressions of equations II-117, II-118, and II-119, it follows that the first three elements of the contribution to the right-hand force vector due to gravity effects are

$$\left\{G_{gg}\right\}_{1,2,3} = g_c \left[\tilde{S}\right] \left\{\gamma_g\right\} - \frac{3g_c}{R_c} \left[\tilde{\gamma}_g\right] \left[J\right] \left\{\gamma_g\right\} \tag{II-125}$$

The second three elements are

$$\left\{G_{gg}\right\}_{4,5,6} = - g_c m \left\{\gamma_g\right\} + \frac{g_c}{R_c} \left(3 \left\{\gamma_g\right\} \left\lfloor\gamma_g\right\rfloor - [I]\right) \left\{S\right\} \tag{II-126}$$

The force due to gravity acting in the k^{th} deformation mode is

$$\begin{aligned} G_{\xi_k} = & - g_c \left\lfloor\gamma_g\right\rfloor \left\{a\right\}_k - \frac{g_c}{R_c} \left[\frac{(b_1)_k + (b_2)_k + (b_3)_k}{2} + e_{kj}\,\xi_j\right] \\ & + \frac{3g_c}{2R_c} \Bigg\{ \left(1 - 2\gamma_{g1}^2\right) \left[(b_1)_k + (C_{11})_{kj}\,\xi_j\right] \\ & \qquad + \left(1 - 2\gamma_{g2}^2\right) \left[(b_2)_k + (C_{22})_{kj}\,\xi_j\right] \\ & \qquad + \left(1 - 2\gamma_{g3}^2\right) \left[(b_3)_k + (C_{33})_{kj}\,\xi_j\right] \\ & + 2\gamma_{g1}\,\gamma_{g2} \left[(b_4)_k + (C_{12})_{kj}\,\xi_j + (C_{12})_{jk}\,\xi_j\right] \\ & + 2\gamma_{g1}\,\gamma_{g3} \left[(b_5)_k + (C_{13})_{kj}\,\xi_j + (C_{13})_{jk}\,\xi_j\right] \\ & + 2\gamma_{g2}\,\gamma_{g3} \left[(b_6)_k + (C_{23})_{kj}\,\xi_j + (C_{23})_{jk}\,\xi_j\right] \Bigg\} \end{aligned} \tag{II-127}$$

where the inertia integrals, $\left(b_n\right)_k$, ($n = 1, 2, \cdots 6$), and $\left(C_{\ell m}\right)_{kj}$, ($\ell, m = 1, 2, 3$), are consistent with the development given in paragraph II.D and Appendix A.

G. Provision for Inclusion of Thermal Environments

All problems associated with thermally induced deflections have in common the requirement that, to determine the effect of solar heating, the spacecraft's attitude relative to the Sun must be known. This required information can be extracted at any point in time from the state vector. It is then necessary to have a model of the response of the flexible structure, either static or dynamic, to solar heating.

Considerable work has been done on modeling flexible appendages in thermal environments (References 2 through 4), and the results indicate that the response depends on the radiation properties of the booms and the attitude relative to the Sun.

The simulation program accounts for time-dependent thermal deformations in the following manner. It is assumed that a model exists whereby the structural deformation of a flexible boom (or appendage) resulting from solar heating can be determined from elements of the state vector and time. This deformation is subtracted from the actual deformation, and the difference is premultiplied by the appendage stiffness matrix. The result is a vector of modified, generalized restoring forces for the appendage, which is summed into the $\{G\}_j$ vector for the appendage body.

In terms of the development given in paragraphs II.B and II.D where $-[k]\{\xi\}$ is seen to be the generalized restoring forces (in the deformation coordinates), note that $-[k]\{\xi\}$ is replaced with $-[k](\{\xi\} - \{\xi_e\})$. The thermal deformation state, $\{\xi_e\}$, is that which must be established from a thermal deformation model.

In this way, a closed-loop response analysis can be achieved, using external subroutines to develop the thermal deformations. Some problems may require only open-loop operation if the variations of $\{\xi_e\}$ in time is slow with respect to general dynamic response.

Rather than building a rigid (or irrevocable) model of thermal deformation, the dynamic simulation program provides the user with an interface whereby he can formulate and code a particular model. Thus, latitude with respect to user requirements is retained.

III. SYNTHESIS AND ANALYSIS OF THE LINEARIZED SYSTEM

Developments to this point have described the analytical techniques used to synthesize the nonlinear characteristics of a dynamical system consisting of an assembly of interconnected flexible (or rigid) bodies. Particular emphasis has been placed on spacecraft systems in which individual bodies that comprise the system may be either spinning or nonspinning and may have large excursions with respect to each other.

This section presents a comprehensive summary of the techniques developed for synthesis and analysis of the linearized dynamic system with particular emphasis on frequency-domain techniques. For the purposes of this discussion, it is convenient to redefine the nature of the system under consideration and, in describing the techniques, to consider the total dynamic system as a *plant* subject to a *controller* rather than as a spacecraft system consisting of interconnected bodies that may be subjected to a control system.

Linearization of the nonlinear state equations is necessary for applying the powerful analytical techniques associated with linear system stability synthesis. When a nonlinear system can be reduced to a linear system in the vicinity of a particular state of interest, it is much more desirable to work with the linearized state equations. An additional feature related to the linearized system permits the analyst to observe linearized perturbation time-response characteristics for the system. The linearized time response can be easily automated by recursive formulas, which are generally more efficient than nonlinear numerical integration algorithms.*

A. Introductory Discussion

The main-line nonlinear time-domain analysis is structured to assemble a collection of interconnected bodies, including a control law. The general form of the governing equations may be concisely indicated as

$$\dot{Y}^i = F\left(Y^i, t\right) \qquad i = 1, 2, \ldots \qquad \text{(III-1)}$$

and the form of the function, F, is the essence of the nonlinear time-domain solution. In fact, it can be stated that equation III-1 is the fundamental basis for the entire DISCOS program. Algorithms for evaluating the nonlinear state-vector time derivatives (and auxiliary equations) are centered in a subprogram and its supporting routines. These functional algorithms are used for linearizing the governing equations about a specified state. In addition, it is desirable to introduce some new variables, including sensor signals, X_{ss}, and control torques, B. These new variables extend the number of equations, and the additional expressions are linearized along with the basic state equations. Additional remarks concerning the use and manipulation of the additional variables are given in a later section. The remainder of this subsection will address specifics relating to the linearization process.

Attention is first focused on a single variable, $\dot{y}_k$, and its dependence on the system state, Y^i, through a known (though possibly nonlinear) functional relationship. Arguments begin by considering an initial system state, Y^i (o), and a functional algorithm with which to evaluate the expression, $\dot{y}_k = d/dt\, y_k$. The unknown, $\dot{y}_k$, is first expressed in terms of a Taylor's series expansion about the given state, Y^i (o), as

$$\dot{y}_k = \dot{y}_k(o) + \frac{\partial \dot{y}_k}{\partial Y^j} dY^j + \frac{\partial^2 \dot{y}_k}{\partial Y^j \partial Y^\ell} dY^j dY^\ell + \ldots \qquad \text{(III-2)}$$

*Reference Paper I provides a broadbrush narrative description of the theoretical development given in this section.

Because the interest here lies in the linear part only, the series is truncated for all partial derivatives greater than one and

$$\dot{y}_k - \dot{y}_k(o) = \frac{\partial \dot{y}_k}{\partial Y^j} dY^j = \dot{y}_{k,j} dY^j \tag{III-3}$$

The task at hand, then, is to establish the partial derivatives indicated as $\dot{y}_{k,j}$, thus yielding an expression (for all $\Delta \dot{Y}^i = \dot{Y}^i - \dot{Y}^i(o)$, i = 1, 2, •••) of the form

$$\Delta \dot{Y}^i = H_{i,j} \Delta Y^j \tag{III-4}$$

Because it would be nearly impossible (certainly impractical) to generalize the determination of the partial derivatives as explicit analytical expressions involving the independent state variables, a numerical approach has been adopted. This task is accomplished by employing numerical perturbation techniques in conjunction with quadratic functions to establish the desired partial derivatives. Symbolically, determination of elements of $H_{i,j}$ is attempted so that

$$\dot{Y}^i = \dot{Y}^i(o) + H_{i,j} \Delta Y^j \tag{III-5}$$

where it is assumed that:

- The functions, $\dot{Y}^i$, are indeed linear, sufficiently near the state, $Y^i(o)$
- The functions, $\dot{Y}^i$, although possibly nonlinear, can be represented as a quadratic (or lower order) in the neighborhood of $Y^i(o)$

The basic approach is concisely summarized in two steps:

- Establish quadratic coefficients for $\dot{Y}^i$ in the vicinity of the state, $Y^i(o)$
- Evaluate the partial derivatives, $H_{i,j}$, at the state, $Y^i(o)$, using the quadratic coefficients and perturbation values on the independent variables.

B. Linearization Process

With reference to figure 8, the quadratic formula can be stated in matrix form as

$$f(\eta) = \lfloor \eta^2 \;\; \eta \;\; 1 \rfloor \begin{Bmatrix} d \\ e \\ f \end{Bmatrix} \tag{III-6}$$

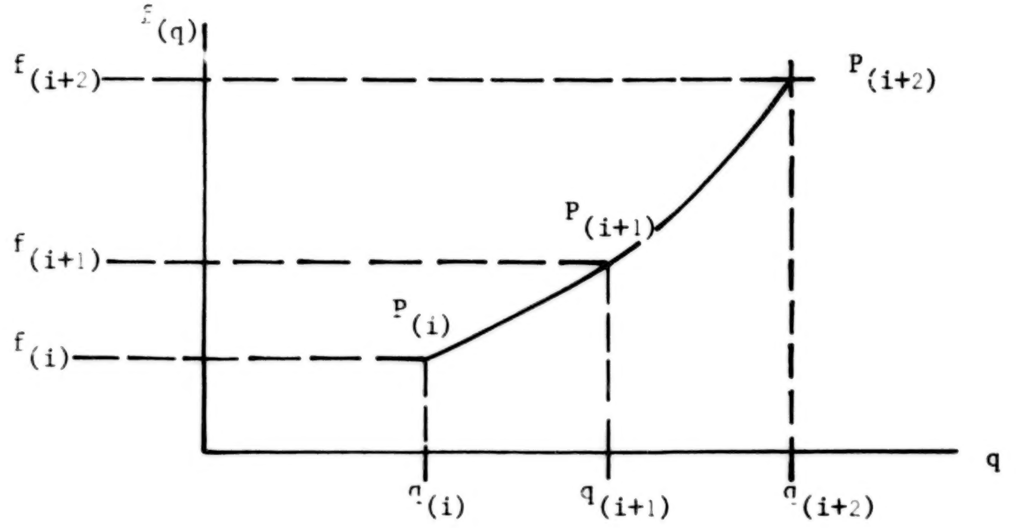


Figure 8. The quadratic formula.

where η is a local spatial coordinate with origin corresponding to $q_{(i)}$, and it is desired to establish the derivative, $\partial f/\partial q$, evaluated at $q_{(i)}$.

In general, the required partial derivative is

$$\frac{\partial f}{\partial q} = \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial q} \tag{III-7}$$

Because the three values, $f_{(i)}$, $f_{(i+1)}$, $f_{(i+2)}$, are evaluated by the previously discussed functional algorithm, these values satisfy equation III-6. More specifically, consider

$$\begin{Bmatrix} f_i \\ f_{i+1} \\ f_{i+2} \end{Bmatrix} = \begin{bmatrix} \eta_i^2 & \eta_i & 1 \\ \eta^2_{i+1} & \eta_{i+1} & 1 \\ \eta^2_{i+2} & \eta_{i+2} & 1 \end{bmatrix} \begin{Bmatrix} d \\ e \\ f \end{Bmatrix} \tag{III-8}$$

and, by matrix manipulation, it follows that

$$f(\eta) = \lfloor \eta^2 \;\; \eta \;\; 1 \rfloor \begin{bmatrix} \eta^2_{i} & \eta_i & 1 \\ \eta^2_{i+1} & \eta_{i+1} & 1 \\ \eta^2_{i+2} & \eta_{i+2} & 1 \end{bmatrix}^{-1} \begin{Bmatrix} f_i \\ f_{i+1} \\ f_{i+2} \end{Bmatrix} \tag{III-9}$$

where the local coordinate, η, is defined as

$$\eta = \frac{q - q_i}{q_{i+2} - q_i} \tag{III-10}$$

and it can be noted that

$$\eta_i = 0; \;\; \eta_{i+2} = 1; \;\; \frac{\partial \eta}{\partial q} = \frac{1}{q_{i+2} - q_i} \tag{III-11}$$

It then follows that

$$f(\eta) = \lfloor \eta^2 \;\; \eta \;\; 1 \rfloor \begin{bmatrix} 0 & 0 & 1 \\ \eta^2_{i+1} & \eta_{i+1} & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} f_i \\ f_{i+1} \\ f_{i+2} \end{Bmatrix} \tag{III-12}$$

and, if $\eta_{i+1} = 1/2$ is specified and it is noted that $f_{(i)} = f_{(\eta = i)}$,

$$f(\eta) = \lfloor \eta^2 \;\; \eta \;\; 1 \rfloor \begin{bmatrix} 0 & 0 & 1 \\ 1/4 & 1/2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} f_{(0)} \\ f_{(1/2)} \\ f_{(1)} \end{Bmatrix} \tag{III-13}$$

$$f(\eta) = \lfloor \eta^2 \;\; \eta \;\; 1 \rfloor \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} f_{(0)} \\ f_{(1/2)} \\ f_{(1)} \end{Bmatrix} \tag{III-14}$$

$$f'(\eta) = \lfloor 2\eta \;\; 1 \;\; 0 \rfloor \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} f_{(0)} \\ f_{(1/2)} \\ f_{(1)} \end{Bmatrix} \tag{III-15}$$

and, in particular,

$$\left.\frac{\partial f}{\partial \eta}\right|_{(0)} = f'_{\eta}(\eta = 0) = e \tag{III-16}$$

and

$$\left.\frac{\partial f}{\partial q}\right|_{(0)} = f'_{q}(q = q_{i}) = \frac{e}{q_{(i+2)} - q_{(i)}} \tag{III-17}$$

1. Comments

Selection of an initial perturbation value, q(i + 2), from an initial specified state, q(o) = Y_k(o), is somewhat arbitrary. A value of 1 percent of the initial value has been successfully used for all example problems during the course of the study. In the case where the initial value is null, an infinitesimal value must be chosen. A value of 1×10^{-5} has been accommodated in the digital code. The intermediate choice of $\eta(i + 1) = 1/2$ was selected for other reasons. Consider first that a single evaluation of a partial derivative, $\partial f/\partial Y^i$ is not sufficient to qualify its validity.

An approach has been employed whereby two successive evaluations of $\partial f/\partial Y^i$ obtained by successively cutting the perturbation in half must agree to a predetermined number of significant digits (e.g., 5).* The choice of $\eta(i + 1) = 1/2$ requires but a single new evaluation

*The establishment of the error criteria that will be used to compare successive evaluations of $\partial f/\partial Y^i$ is strongly dependent on computer word size. In special cases, numerical noise can exceed the established error criteria and prevent numerical convergence. This problem can usually be circumvented by changing error criteria limits set in subroutine LINEAR.

for each element in $\dot{Y}^i$ at each successive reduction in the perturbation value. In summary, the linearization employs an iterative technique to establish the desired partial derivatives.

2. System Resonance Properties

The linearization process has provided a system of first-order differential equations that describe the dynamical simulation in terms of perturbation variables about an equilibrium state. The linearized canonical form appears as

$$\Delta\dot{Y}^i = H_{i,j}\Delta Y^j \qquad (i,j = 1,2,\ldots) \tag{III-18}$$

The coefficients, $H_{i,j}$, contain all of the resonance frequency properties of the dynamical system. The standard eigensolution form is indicated by taking the transform of this expression to obtain

$$\left(\delta_i^{\,j} s - H_{i,j}\right)\Delta Y^j(s) = 0 \tag{III-19}$$

Extraction of the roots (eigenvalues) from $H_{i,j}$ then gives the roots of the dynamical system. There will be N of these roots, and any complex roots will appear as conjugate pairs because the elements of $H_{i,j}$ are all real. The imaginary part of the complex pairs represents the resonance (or characteristic) frequencies of the system.

C. Exchange of Variables

It is often necessary for the analyst to require additional variables with which to assess the stability characteristics of the dynamical system. These additional variables ordinarily take the form of plant sensor signals and control system output forces and torques. Although the desired variables may not be explicitly contained in the system state vector, Y^i, they are known in terms of the state variables through an expression of the form

$$w^j = g(Y^i) \tag{III-20}$$

Recall also from previous discussions that it has been established either directly or through linearization that

$$\Delta\dot{Y}^i = H_{i,j}\,\Delta Y^j \tag{III-21}$$

Now, rewriting equation III-20 in matrix form and identifying variables to retain, Y_1, and variables to eliminate, Y_2, gives

$$\{w\} = \left[C_1 \,\vdots\, C_2\right] \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} \tag{III-22}$$

and it can readily be established that

$$\{Y\} = [R]\,\{z\}$$

where

$$[R] = \left[\begin{array}{c|c} I & 0 \\ \hline -C_2^{-1}\,C_1 & C_2^{-1} \end{array}\right] \tag{III-23}$$

and

$$\{z\} = \begin{Bmatrix} Y_1 \\ w \end{Bmatrix}$$

Thus, the state equations for the dynamical system can be written (in terms of variables that include the desired plant sensor signals and control system forces and torques) as

$$\{\dot{z}\} = [R]^{-1}\left[H_{i,j}\right][R]\,\{z\} \tag{III-24}$$

and the transformation $A_{ij} = R^{-1}\,H_{i,j}\,R$, is commonly referred to as a similarity transformation. The matrix, A_{ij}, is said to be the transform of $H_{i,j}$ by the matrix R (Reference 5).

The similarity transformation, A_{ij}, possesses a unique property in that the eigenvalues of A_{ij} are equal to the eigenvalues of $H_{i,j}$! A simple proof that establishes this point follows:

The characteristic matrix of A_{ij} is given by

$$(A_{ij} - sI) = (R^{-1}\,H_{i,j}\,R - sI) = R^{-1}\,(H_{i,j} - sI)\,R \tag{III-25}$$

It follows that $Q(s)$, the characteristic polynomial of A_{ij}, is

$$Q(s) = \det(A_{ij} - sI) = \det R^{-1} (\det(H_{i,j} - sI)) \det R$$

and, as $(\det R^{-1}) = 1/\det(R)$, it is apparent that

$$Q(s) = \det(H_{i,j} - sI) = P(s)$$

where $P(s)$ is the characteristic polynomial of $H_{i,j}$. Thus, it is evident that the matrices, $H_{i,j}$ and A_{ij}, have the same characteristic equations

$$Q(s) = P(s) = 0$$

and, therefore, the eigenvalues of A_{ij} are equal to the eigenvalues of $H_{i,j}$.

Application of this property now permits isolation of the plant and controller, even for a state-space representation of an inherently nonlinear system that can be linearized about a specified state. Separation of plant and control-system variables is an important facet of linear system stability synthesis.

1. Evaluation of the Similarity Transformation

This discussion relates to a procedural approach for determining the similarity transformation matrix, $[R]$, that will relieve the user from the burden of having to select those variables to eliminate from the original state vector so that the auxiliary variables, B^i and $X_{ss}{}^i$, can become an independent constituent of the modified state vector for use in the linearized studies. With reference to equation III-22, all the C_{ij} coefficients are known because they have been obtained through linearization of the auxiliary equations. The C_{ij} coefficients simply define the dependence of the auxiliary variables, w^j, on the original state variables, Y^i. In general, it is not possible to directly partition the C_{ij} in the C_1 and C_2 partitions as indicated in equation III-22, for the decision has not been made yet as to which state variables to retain and which to discard in preference to introducing the auxiliary variables, w^j. In this light, a *best possible* choice is made with regard to which of the variables to eliminate from the state vector, Y^i, so that the auxiliary variables, w^j, may be included. A one to one variable exchange will often occur between an element of w^j and an element of Y^i. In any case, a variable exchange is necessary for structuring the total system into the desired plant/controller framework whereby the plant and controller can be isolated along with the plant sensor signals and the control-system inputs.

The following approach is used in this simulation to accomplish the desired result (namely, an optimum selection from Y^i as to which variables to eliminate so that w^j can be introduced as a part of the state vector). With reference to equation III-22,

$$\left[C \mid -I\right] \begin{Bmatrix} Y^i \\ w^j \end{Bmatrix} = \{0\} \qquad \text{(III-26)}$$

The primary focus of attention is now directed to a systematic examination of the C_{ij} coefficients so that the variable exchange is accomplished in an optimum manner. To help clarify the discussion, some size identifications are noted:

C_{ij} has size NR by NS

Y^i has size NS by 1

w^j has size NR by 1

NJQ = NS + NR

Clearly, at least one nonzero element exists in each row of the C_{ij} array. Otherwise, Y^i does not represent an independent set.

Now, a search through the first NS elements of row 1 in the matrix array

$$\left[C \mid -I\right] \qquad \text{(III-27)}$$

will identify the largest element (absolute value) in row 1. Assuming that this element occurs in column JBIG ($1 \leq \text{JBIG} \leq \text{NS}$) permits the division of each element of row 1 by this largest element, and subsequent elementary row operations on rows 1 through NR will eliminate those elements below the pivotal element in column JBIG. This procedure is repeated for each of the NR rows contained in the matrix, and the following observations are noted:

- The appearance of a one (1.0) in a row identifies a variable that will be eliminated in preference to including an element of w^j.
- The absence of a zero or one in columns of a given row indicates which variables will survive the exchange process.
- All variables in w^j (NR of them) will become part of a new and independent state vector (the modified state vector).
- The transformation, R_{ij} (i, j = 1 . . . NS) can be constructed from the matrix that remains after the procedural approach has exhausted all of the NR rows of expression III-27.

2. Illustrative Example

A simple example is presented to further describe the actual mechanics used for evaluating the similarity transformation. Linearization of the auxiliary equations has established that

$$\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = [C] \begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{Bmatrix}$$

and, to implement the proposed algorithm, the C matrix augmented with -I is first written and the elements of C are further identified as

$$\left[\overset{NS}{C} \;\middle|\; \overset{2}{-I}\right] = \left[\begin{array}{cccc|cc} b_{11} & b_{12} & b_{13} & b_{14} & -1 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & 0 & -1 \end{array}\right]$$

For illustrative purposes, assume that $|b_{13}| > |(b_{11}, b_{12}, b_{14})|$, and, hence, b_{13} is the pivotal element for the first row. It follows that row 2 is modified by the relation

$$ele_{2j} = -b_{23}\frac{b_{1j}}{b_{13}} + b_{2j}$$

giving the matrix

$$\left[\begin{array}{c|c|c|c|c|c} \dfrac{b_{11}}{b_{13}} & \dfrac{b_{12}}{b_{13}} & 1 & \dfrac{b_{14}}{b_{13}} & -\dfrac{1}{b_{13}} & 0 \\ \hline -b_{23}\dfrac{b_{11}}{b_{13}} + b_{21} & -b_{23}\dfrac{b_{12}}{b_{13}} + b_{22} & 0 & -b_{23}\dfrac{b_{14}}{b_{13}} + b_{24} & \dfrac{b_{23}}{b_{13}} & -1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c|c} c_{11} & c_{12} & 1 & c_{14} & c_{15} & 0 \\ \hline c_{21} & c_{22} & 0 & c_{24} & c_{25} & -1 \end{array}\right]$$

The process is continued by first dividing row 2 by the largest element and again using row operations to eliminate the other element in that column. Again, for illustrative purposes, it is assumed that the pivotal element of row 2 (say c_{22}) is known, and row 1 will be modified by row operations as

$$ele_{1j} = - c_{12} \frac{c_{2j}}{c_{22}} + c_{1j}$$

giving

$$d_{ij} = \left[\begin{array}{c|c|c|c|c|c} -c_{12}\dfrac{c_{21}}{c_{22}} + c_{11} & 0 & 1 & -c_{12}\dfrac{c_{24}}{c_{22}} + c_{14} & -c_{12}\dfrac{c_{25}}{c_{22}} + c_{15} & \dfrac{c_{12}}{c_{22}} \\ \hline \dfrac{c_{21}}{c_{22}} & 1 & 0 & \dfrac{c_{24}}{c_{22}} & \dfrac{c_{25}}{c_{22}} & -\dfrac{1}{c_{22}} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c|c} d_{11} & 0 & 1 & d_{14} & d_{15} & d_{16} \\ \hline d_{21} & 1 & 0 & d_{24} & d_{25} & d_{26} \end{array}\right]$$

from which the desired similarity transformation is established as

$$R_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -d_{21} & -d_{24} & -d_{25} & -d_{26} \\ -d_{11} & -d_{14} & -d_{15} & -d_{16} \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

It now follows that the original state variables are written in terms of the modified state variables as

$$\begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -d_{21} & -d_{24} & -d_{25} & -d_{26} \\ -d_{11} & -d_{14} & -d_{15} & -d_{16} \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_4 \\ w_1 \\ w_2 \end{Bmatrix}$$

D. System Transfer Functions

The entire system transfer function synthesis can be concisely summarized in a chronological sequence of steps that began with linearization of the coupled mechanical/control law equations that govern the dynamical motion. This process included linearization of additional equations that contained specific variables required for further consideration in the stability analysis (namely, plant sensor signals and control system outputs). A similarity transformation has been introduced that exchanges original state variables for these desired sensor signals and controller outputs so that the resulting modified state vector is still representative of an independent set of state variables. The resulting system of state-space equations is later identified as equation III-28.

The system characteristic matrix, A_{ij}, provides the basis for evaluating the coupled mechanical/control system resonant characteristics (natural frequencies), as well as the fundamental basis for specification and determination of the various types of transfer functions. The next subsection addresses some of the more specific details regarding specific transfer function relationships. A particular transfer function is identified by a *type* along with the desired output/input variable designation. An eigenvalue problem is then stated, which leads to determination of the numerator roots (zeros) and denominator roots (poles) for the particular transfer function. When the poles and zeros are known for a transfer function, this information can be further processed and displayed by any of the conventional display modes: Bode, Nichols, Nyquist, and/or root locus.

The conventional block diagram representation for the coupled plant/controller system (figure 9) provides additional insight for determination of system transfer functions.

The first-order differential equations for the system are written as

$$\dot{Z}^i = A_{ij} Z^j + B_{T_{ij}} R_T^{\,j} + B_{s_{ij}} R_s^{\,j} \qquad \text{(III-28)}$$

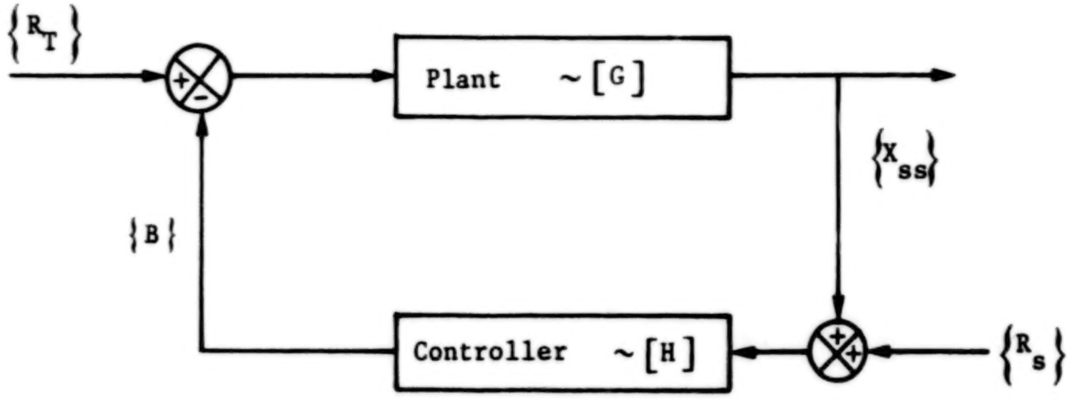


Figure 9. Plant/controller block diagram.

and it is helpful at this point to express the equation in matrix form and indicate the separate partitioned subsets of $\dot{Z}^i$, A_{ij}, Z^j, $B_{T_{ij}}$, $R_T^{\,j}$, $B_{s_{ij}}$, and $R_s^{\,j}$ as

$$\begin{Bmatrix} \dot{y} \\ \dot{X}_{ss} \\ \dot{\delta} \\ \dot{B} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} y \\ X_{ss} \\ \delta \\ B \end{Bmatrix} + \begin{bmatrix} b_T{}^1 \\ b_T{}^2 \\ b_T{}^3 \\ b_T{}^4 \end{bmatrix} \{R_T\} + \begin{bmatrix} b_s{}^1 \\ b_s{}^2 \\ b_s{}^3 \\ b_s{}^4 \end{bmatrix} \{R_s\} \qquad \text{(III-29)}$$

The following observations are noted

$$\begin{array}{lll} a_{31} = 0 & b_T{}^1 = -a_{14} & b_s{}^1 = 0 \\ a_{41} = 0 & b_T{}^2 = -a_{24} & b_s{}^2 = 0 \\ a_{13} = 0 & b_T{}^3 = 0 & b_s{}^3 = a_{32} \\ a_{23} = 0 & b_T{}^4 = 0 & b_s{}^4 = a_{42} \end{array}$$

and equation III-29 can be restated as

$$\begin{Bmatrix} \dot{y} \\ \dot{X}_{ss} \\ \dot{\delta} \\ \dot{B} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & & a_{14} \\ a_{21} & a_{22} & & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} y \\ X_{ss} \\ \delta \\ B \end{Bmatrix} + \begin{bmatrix} -a_{14} \\ -a_{24} \\ 0 \\ 0 \end{bmatrix} \left\{ R_T \right\} + \begin{bmatrix} 0 \\ 0 \\ a_{32} \\ a_{42} \end{bmatrix} \left\{ R_s \right\} \qquad \text{(III-30)}$$

Equation III-30 is the operating basis for stating particular transfer function relationships for the plant/controller system.

The general procedure is to establish a system transfer function between inputs R_T and R_s and outputs X_{ss} and B. Loops may be opened to provide open-loop information by manipulating the A_{ij} coefficients to prohibit certain feedbacks.

To symbolically describe specification of a transfer function, begin by consolidating the b coefficients and taking the Laplace transform of equation III-30 to give

$$\left[Is \right] \left\{ Z(s) \right\} = \left[A \right] \left\{ Z(s) \right\} + \left[b \right] \left\{ U(s) \right\} \qquad \text{(III-31)}$$

or

$$\left[\left[Is \right] - \left[A \right] \right] \left\{ Z(s) \right\} = \left[b \right] \left\{ U(s) \right\} \qquad \text{(III-32)}$$

and then employ Cramer's rule to evaluate a given element, $Z(s)^p$, due to a particular input, $U(s)^q$, where

$$Z(s)^p / U(s)^q = \frac{\text{aug}\,|Is - A|}{|Is - A|} \qquad \text{(III-33)}$$

and where aug $|Is - A|$ is accomplished by placing column q of b into column p of $|Is - A|$

The Q-R algorithm (References 6 and 7) is a useful tool with which to extract the indicated determinants in equation III-33.

1. Root Extraction Process

With reference to equation III-33, evaluation of both the numerator and denominator roots is desired. The denominator root extraction is straightforward in that $p_1, p_2, p_3, \cdots p_n$ is sought from an expression of the form

$$D(s) = \det([I]s - [A])$$

so that

$$D(s) = (s - p_1)(s - p_2) \cdots (s - p_n) = \prod_{i=1}^{n} (s - p_i) \qquad \text{(III-34)}$$

This evaluation is completed by extracting the characteristic roots of the matrix, A_{ij}. In general, these roots will be complex because A_{ij} is not symmetric.

The process used for evaluating the numerator is best illustrated with an example. Consider that we have the (4 by 4) characteristic system matrix,

$$[A_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

and the column of coefficients b_i that premultiply the desired input variable, U^q. Further, let it be desired to obtain the transfer function relating output of the third variable in the state equations, y_3, to the input, U^q.

The state equations for this system would appear as

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} + \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix} U^q \qquad \text{(III-35)}$$

and, with reference to equation III-33, the numerator is

$$N(s) = \text{aug}\, |Is - A|$$

or

$$N(s) = \det \begin{vmatrix} s - a_{11} & -a_{12} & b_1 & -a_{14} \\ -a_{21} & s - a_{22} & b_2 & -a_{24} \\ -a_{31} & -a_{32} & b_3 & -a_{34} \\ -a_{41} & -a_{42} & b_4 & s - a_{44} \end{vmatrix} \tag{III-36}$$

After performing elementary row operations, equation III-36 can be restated in the form

$$N(s) = b_3 \det \begin{vmatrix} s - a_{11} + a_{31} b_1/b_3 & -a_{12} + a_{32} b_1/b_3 & -a_{14} + a_{34} b_1/b_3 \\ -a_{21} + a_{31} b_2/b_3 & s - a_{22} + a_{32} b_2/b_3 & -a_{24} + a_{34} b_2/b_3 \\ -a_{41} + a_{31} b_4/b_3 & -a_{42} + a_{32} b_4/b_3 & s - a_{44} + a_{34} b_4/b_3 \end{vmatrix} \tag{III-37}$$

or, in symbolic terms, as

$$N(s) = b_3 \det |\, [Is] - [\tilde{a}]\, | \tag{III-38}$$

where the matrix $\tilde{a}$ is given as

$$\begin{bmatrix} a_{11} - a_{31} b_1/b_3 & a_{12} - a_{32} b_1/b_3 & a_{14} - a_{34} b_1/b_3 \\ a_{21} - a_{31} b_2/b_3 & a_{22} - a_{32} b_2/b_3 & a_{24} - a_{34} b_2/b_3 \\ a_{41} - a_{31} b_4/b_3 & a_{42} - a_{32} b_4/b_3 & a_{44} - a_{34} b_4/b_3 \end{bmatrix}$$

Note that the previous expression for N (s) is finite only if $b_3 \neq 0$, and the question is: Can b_3 realistically be null? The answer is *yes*, as the following example indicates.

Consider the simple mechanical system consisting of two masses connected by a single spring/ dashpot combination as shown in figure 10.

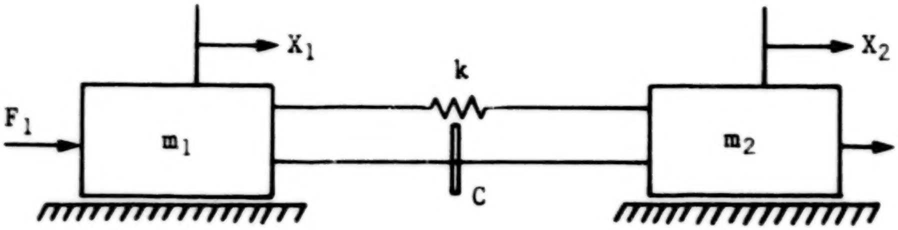


Figure 10. Simple mechanical system.

The state-space representation is

$$\frac{d}{dt}\begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ X_1 \\ X_2 \end{Bmatrix} = \left[\begin{array}{c|c|c|c} -c/m_1 & c/m_1 & -k/m_1 & k/m_1 \\ c/m_2 & -c/m_2 & k/m_2 & -k/m_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ X_1 \\ X_2 \end{Bmatrix} + \begin{bmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

and the frequency domain (or transformed) equations in s are

$$\Big[[I]s - [A]\Big]\begin{Bmatrix} \dot{X}_1(s) \\ \dot{X}_2(s) \\ X_1(s) \\ X_2(s) \end{Bmatrix} = \begin{bmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

where

$$[A] = \begin{bmatrix} -c/m_1 & c/m_1 & -k/m_1 & k/m_1 \\ c/m_2 & -c/m_2 & k/m_2 & -k/m_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now, consider the transfer function, $\dot{X}_1(s)/F_1$, where the augmented numerator is

$$N(s) = \det \begin{vmatrix} 1/m_1 & -c/m_1 & k/m_1 & -k/m_1 \\ 0 & s + c/m_2 & -k/m_2 & k/m_2 \\ 0 & 0 & s & 0 \\ 0 & -1 & 0 & s \end{vmatrix}$$

and the pivot element is the (1, 1) element or $1/m_1 \neq 0$. On the other hand, the transfer function, $X_1\ (s)/F_1$, has the augmented numerator

$$N(s) = \det \begin{vmatrix} s + c/m_1 & -c/m_1 & 1/m_1 & -k/m_1 \\ -c/m_2 & s + c/m_2 & 0 & k/m_2 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & s \end{vmatrix}$$

and the pivot element is the (3, 3) element, which is null.

The problem to be addressed now involves evaluation of the numerator determinant, N(s), when the pivotal element is null. The particular mathematical problem may be restated as

$$N(s) = \det \left| [\tilde{I}]s - [\tilde{A}] \right| \tag{III-39}$$

where $[\tilde{I}]$ is the identity matrix, $[I]$, of size N with a null diagonal element. Addition and subtraction of the quantity, $[\tilde{I}]\,\chi$, where χ is an arbitrary constant not equal to one of the roots of $[\tilde{A}]$, yields

$$N(s) = \det \left| [\tilde{I}]\,(s - \chi) - \left[[\tilde{A}] - [\tilde{I}]\,\chi \right] \right| \qquad \text{(III-40)}$$

and, if $(s - \chi) \equiv 1/p$ is defined, there results

$$N(s) = \frac{(-1)^N}{p^N} \det \left| \tilde{A} - \tilde{I}\chi \right| \left(\det \left| Ip - (\tilde{A} - \tilde{I}\chi)^{-1}\tilde{I} \right| \right) \qquad \text{(III-41)}$$

The roots, $(p_i, i = 1, N)$, are found as the eigenvalues of the expression

$$\left[[\tilde{A}] - [\tilde{I}]\,\chi \right]^{-1} [\tilde{I}] \qquad \text{(III-42)}$$

and the eigensolution permits N(s) to be written as

$$N(s) = \frac{(-1)^N}{p^N} \det \left| \tilde{A} - \tilde{I}\chi \right| \left\{ (p - p_1)(p - p_2) \cdots (p - p_N) \right\} . \qquad \text{(III-43)}$$

The following observation can now be made: a p_i equal to zero implies a root at infinity (or a characteristic polynomial having an order less than N). Thus, the null p_i's are eliminated from the expression, giving the characteristic polynomial an order, n, which is less than N. It is a rather common occurrence for the number of zeros (an order of N(s) to be significantly less than the number of poles (order of D(s)). With reference to equation III-43, the numerator expression, N(s), can be written as

$$N(s) = (-1)^N \det \left| \tilde{A} - \tilde{I}\chi \right| \left\{ (1 - \frac{p_1}{p})(1 - \frac{p_2}{p}) \cdots (1 - \frac{p_n}{p}) \right\} \qquad \text{(III-44)}$$

and, recalling that $p = 1/s\text{-}\chi$, yields

$$N(s) = \frac{(-1)^N}{\prod_{i=1}^{n} (\chi - S_i)} \det \left| \tilde{A} - \tilde{I}\chi \right| \left\{ (s - s_1)(s - s_2) \cdots (s - s_n) \right\} \qquad \text{(III-45)}$$

Next, note that

$$\prod_{i=1}^{n} (\chi - s_i) = \prod_{i=1}^{n} \left(\frac{-1}{p_i} \right) \qquad \text{(III-46)}$$

and it follows that

$$N(s) = (-1)^{N-n} \prod_{i=1}^{n} p_i \det \left| \tilde{A} - \tilde{I}\chi \right| \prod_{i=1}^{n} (s - s_i) \qquad \text{(III-47)}$$

The numerator root gain, k_R, can now be identified as

$$k_R = (-1)^{N-n} \prod_{i=1}^{n} p_i \det \left| \tilde{A} - \tilde{I}\chi \right| \qquad \text{(III-48)}$$

and the Bode gain, k_B, for the numerator is

$$k_B = k_R (-1)^m \prod_{i=1}^{m} s_i$$

where $m \leqslant n$

2. Transfer-Function Classification

With reference to figure 9, it is possible to directly identify six transfer function types, each characterizied by the specific variables involved and by the presence of feedback. In addition, a seventh type will also be described whereby certain control variables feed back and others do not. This type is similar to an open-loop transfer function but treats selected channels of the controller as part of the mechanical system (plant). During the course of this discussion,

it will become apparent that additional transfer function types are easily accommodated by rather simple manipulations with the system characteristic matrix, A_{ij}.*

In general, note that the process of obtaining the desired transfer function involves but a few basic steps. The transfer function characteristic matrix, $\mathbb{R}_{ij}$, and the desired force coefficient vector, b_i, are obtained directly from the system characteristic matrix, A_{ij}. These two matrices are then put in a form so that the Q-R algorithm can be used to extract system roots.

a. *Type I (Plant Only)*—Type I is the forward path transfer function for the plant with no feedback and is of the form

$$X_{ss}^{P}/R_T^q = G(s) \tag{III-49}$$

The control variables, δ^i, and control outputs, B^i, do not feed back into the plant. The matrix expression depicting the system of interest is

$$\frac{d}{dt}\begin{Bmatrix} y \\ X_{ss} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{Bmatrix} y \\ X_{ss} \end{Bmatrix} + \begin{bmatrix} b_{T}1 \\ b_{T}2 \end{bmatrix}\begin{Bmatrix} R_T \end{Bmatrix} \tag{III-50}$$

The matrix, A_{ij}, to use in the general expression given as equation III-33 is referred to as $\mathbb{R}_{ij}$ or the reduced A_{ij} matrix,

$$\mathbb{R}_{ij} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{III-51}$$

The augmented $\mathbb{R}_{ij}$ matrix is obtained by removing the column corresponding to the input variable, R_T^q, from the expression, b_T, and inserting this column into the column in $\mathbb{R}_{ij}$, which corresponds to the desired output, X_{ss}^P. The resulting transfer function is then given as

$$X_{ss}^{P}/R_T^q = \frac{\text{aug } | Is - \mathbb{R}|}{| Is - \mathbb{R}|} \tag{III-52}$$

*Additional comments pertaining to the exact definitions of various types of transfer functions implemented can be found in subroutine DEF5 Debug 116 in the discussion concerning "Frequency Domain Analysis" (Volume II, Appendix B, page B-332).

b. Type II (Controller Only)—Type II represents the feedback path, H(s), for the controller only. The desired transfer function relates control-system outputs, B^i, to sensor signal inputs, X^j_{ss},

$$B^p/X^q_{ss} = H(s) \qquad \text{(III-53)}$$

The reduced characteristic matrix, $\mathbb{R}_{ij}$, and the corresponding input coefficients, b_{ik}, are given as

$$\mathbb{R}_{ij} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}, \quad b_{ik} = \begin{bmatrix} a_{32} \\ a_{42} \end{bmatrix} \qquad \text{(III-54)}$$

c. Type III (Open Loop, HG)—Type III falls within the framework of the classical open-loop transfer function designation and relates control system outputs, B^i, to external plant inputs, R^j_T. The algebraic expression for a given output variable, B^p, due to an external input, R^q_T, is indicated as

$$B^p/R^q_T = (HG)(s) \qquad \text{(III-55)}$$

The open-loop system characteristic matrix, $\mathbb{R}_{ij}$, and corresponding input coefficients, b_{ik}, are

$$\mathbb{R}_{ij} = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & & \\ \hline a_{21} & a_{22} & & \\ \hline & a_{32} & a_{33} & a_{43} \\ \hline & a_{42} & a_{43} & a_{44} \end{array}\right], \quad b_{ik} = \begin{bmatrix} -a_{14} \\ -a_{24} \\ 0 \\ 0 \end{bmatrix} \qquad \text{(III-56)}$$

It was previously noted that $a_{31} = a_{41} = a_{13} = a_{23} = 0$, and, in addition, the partitions, a_{14} and a_{24}, are set to zero to prohibit the B^i feedback. Thus, the loop is opened to establish HG, and the open-loop transfer function in s. Note that the negative sign in the b_{ik} coefficients simply indicates that the B^i feedback is negative with respect to the external plant inputs, R^j_T.

d. Type IV (Open Loop, GH)—An additional open-loop transfer function is often desired for assessing the plant sensor signal outputs caused by controller noise inputs. The transfer function then relates sensor signal outputs, X_{ss}^{i}, to control system noise inputs, R_s^{j}. The plant sensor signal vector does not feed back into the system so that

$$X_{ss}^{p}/R_s^{q} = (GH)(s) \tag{III-57}$$

and the system characteristic matrix, $\mathbb{R}_{ij}$, and the external input coefficients, b_{ik}, are identified as

$$\mathbb{R}_{ij} = \begin{bmatrix} a_{11} & a_{12} & & a_{14} \\ a_{21} & a_{22} & & a_{24} \\ & & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{bmatrix}, \quad b_{ik} = \begin{bmatrix} 0 \\ 0 \\ a_{32} \\ a_{42} \end{bmatrix} \tag{III-58}$$

Note that the a_{32} and a_{42} partitions have been nulled to eliminate sensor signal feedback.

e. Type V (Closed Loop – Control Ratio)—The system control ratio is given as the transfer function that relates plant variable outputs to externally applied plant inputs with the control system entirely active. This transfer function is expressed as

$$X_{ss}^{p}/R_T^{q} = \left(\frac{G}{1+GH}\right)(s) \tag{III-59}$$

and the system characteristic matrix, $\mathbb{R}_{ij}$, and the external input coefficients, b_{ik}, are identified as

$$\mathbb{R}_{ij} = \begin{bmatrix} a_{11} & a_{12} & & a_{14} \\ a_{21} & a_{22} & & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad b_{ik} = \begin{bmatrix} -a_{14} \\ -a_{24} \\ 0 \\ 0 \end{bmatrix} \tag{III-60}$$

The negative sign in the matrix, b_{ik}, indicates that the feedback is negative.

f. Type VI (Closed Loop)–An additional closed-loop transfer function has been accommodated within the digital simulation. Specifically, Type VI relates plant sensor signal outputs to sensor signal noise inputs with all control-system loops active. The transfer function is symbolically indicated as

$$X_{ss}^{p} = (\text{transfer function})\, R_{s}^{q} \tag{III-61}$$

where the system characteristic matrix, $\mathbb{R}_{ij}$, and corresponding input coefficients are identified as

$$\mathbb{R}_{ij} = \begin{bmatrix} a_{11} & a_{12} & & a_{14} \\ a_{21} & a_{22} & & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad b_{ik} = \begin{bmatrix} 0 \\ 0 \\ a_{32} \\ a_{42} \end{bmatrix} \tag{III-62}$$

g. Type VII (Quasi-Open Loop)–An additional transfer function type is identified here and is referred to as a quasi-open loop (figure 11). It is of the open-loop type in that we are interested in control-system outputs, B^i, attributable to plant variable inputs, R_T^j. For example, suppose that, for a multichannel control system (such as azimuth and elevation), outputs B^i are desired on the controller channel that do not feed back and suppose that the other channel is active in that it feeds back into the plant.

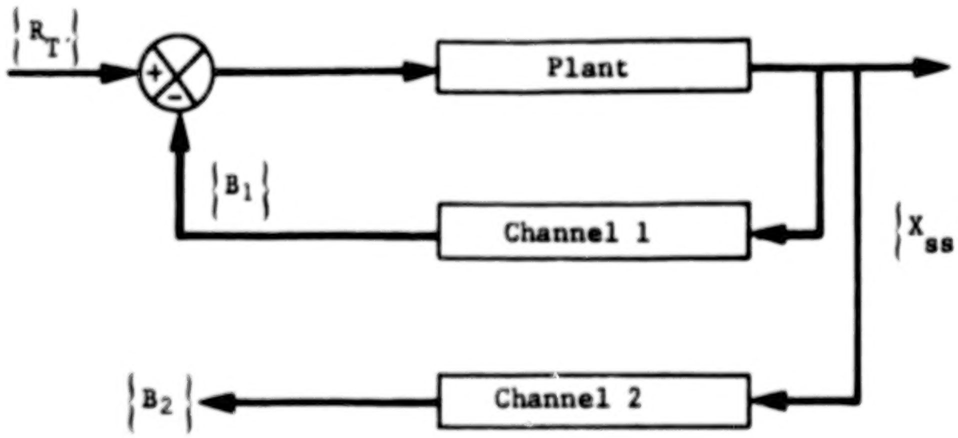


Figure 11. Type VII quasi-open loop block diagram.

For the configuration indicated, a typical Type VII transfer function (TF) would be given by

$$B_2^p = \text{(transfer function)}\ R_T^q$$

and the form of the system characteristic matrix, $\mathbb{R}_{ij}$, and plant input coefficient matrix, b_{ik}, would be

$$\mathbb{R}_{ij} = \begin{bmatrix} a_{11} & a_{12} & & \tilde{a}_{14} \\ a_{21} & a_{22} & & \tilde{a}_{24} \\ & a_{32} & a_{33} & a_{34} \\ & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad b_{ik} = \begin{bmatrix} -a_{14} \\ -a_{24} \\ 0 \\ 0 \end{bmatrix} \qquad \text{(III-63)}$$

The subpartitions, $\tilde{a}_{14}$ and $\tilde{a}_{24}$, indicate modification of the original partitions, a_{14} and a_{24}. Specifically, $\tilde{a}_{mn}$ is a subset of a_{ij} that is obtained by keeping only those n columns of a_{mn} that correspond to the B^i variables that feed back to the plant.

3. Transfer Functions—Polynomial Description

This subsection is addressed to implementation of control-system transfer functions described as the ratio of two polynomials in the frequency domain, s. Specifically, consider

$$TF = P(s)/Q(s) \qquad \text{(III-64)}$$

where

$$Q(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots + a_n{}^n$$

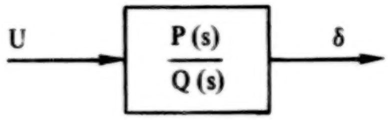
and

$$P(s) = b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m$$

Because the previously described governing equations have been stated in canonical first-order form, the polynomial description for the transfer function is restated in the form

$$\dot{\delta}^i = A_{ij}\,\delta^j + B_i U \qquad \text{(III-65)}$$

The block diagram for the system is



from which we write

$$\delta = \frac{P(s)}{Q(s)}\,U \qquad \text{(III-66)}$$

and expansion of the implied operator in s results in a differential equation of the form

$$a_n \overset{n}{\delta} + a_{n-1} \overset{n-1}{\delta} + \cdots + a_1 \dot{\delta} + a_0 \delta = b_m \overset{m}{U} + b_{m-1} \overset{m-1}{U} + \cdots + b_1 \dot{U} + b_0 U \qquad \text{(III-67)}$$

where

$$\overset{n}{\delta} = \frac{d^n \delta}{dt^n}$$

In general, the order of P(s) will be no greater than the order of Q(s) or $m \leq n$.

a. $m = n$ – Equation III-67 is divided by a_n to obtain

$$\overset{n}{\delta} + C_{n-1} \overset{n-1}{\delta} + \cdots + C_1 \dot{\delta} + C_0 \delta = d_m \overset{m}{U} + d_{m-1} \overset{m-1}{U} + \cdots + d_1 \dot{U} + d_0 U \qquad \text{(III-68)}$$

where

$$C_i = \frac{a_i}{a_n}$$

and

$$d_i = \frac{b_i}{a_n}$$

The following example is used for illustration:

Consider the equation with m = n = 4,

$$\ddddot{\delta} + C_3 \dddot{\delta} + C_2 \ddot{\delta} + C_1 \dot{\delta} + C_0 \delta = d_4 \ddddot{U} + d_3 \dddot{U} + d_2 \ddot{U} + d_1 \dot{U} + d_0 U$$

or, in operator form,

$$s^4 \delta + s^3 C_3 \delta + s^2 C_2 \delta + s C_1 \delta + C_0 \delta = s^4 d_4 U + s^3 d_3 U + s^2 d_2 U + s d_1 U + d_0 U$$

This can be rewritten as

$$s^4 \left(\delta - d_4 U\right) + s^3 \left(C_3 \delta - d_3 U\right) + s^2 \left(C_2 \delta - d_2 U\right) + s \left(C_1 \delta - d_1 U\right) + \left(C_0 \delta - d_0 U\right) = 0$$

and the substitution

$$\delta_1 = \delta - d_4 U$$

permits a reduction in order to

$$s^3 \left(\dot{\delta}_1 + C_3 \delta - d_3 U\right) + s^2 \left(C_2 \delta - d_2 U\right) + s \left(C_1 \delta - d_1 U\right) + \left(C_0 \delta - d_0 U\right) = 0$$

A new variable can again be introduced:

$$\delta_2 = \left(\dot{\delta}_1 + C_3 \delta - d_3 U\right)$$

and the foregoing can be rewritten as

$$s^2(\dot{\delta}_2 + C_2\delta - d_2U) + s(C_1\delta - d_1U) + (C_0\delta - d_0U) = 0$$

It follows that, if

$$\delta_3 = \dot{\delta}_2 + C_2\delta - d_2U$$

is defined,

$$s(\dot{\delta}_3 + C_1\delta - d_1U) + C_0\delta - d_0U = 0$$

results, and the substitution

$$\delta_4 = \dot{\delta}_3 + C_1\delta - d_1U$$

gives

$$\dot{\delta}_4 = -C_0\delta + d_0U$$

The variable, δ, can now be eliminated from each of the foregoing expressions, and the results can be generalized to n^{th}-order systems.

The result is concisely stated as a matrix equation that is recognized to be of the desired form initially given as equation III-65.

$$\begin{Bmatrix} \dot{\delta}_1 \\ \dot{\delta}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{\delta}_{n-1} \\ \dot{\delta}_n \end{Bmatrix} = \begin{bmatrix} -C_{n-1} & 1 & 0 & \cdot & \cdot & 0 \\ -C_{n-2} & 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ -C_1 & 0 & 0 & & & 1 \\ -C_0 & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \delta_{n-1} \\ \delta_n \end{Bmatrix} + \begin{Bmatrix} d_{n-1} - C_{n-1}d_n \\ d_{n-2} - C_{n-2}d_n \\ \cdot \\ \cdot \\ \cdot \\ d_1 - C_1d_n \\ d_0 - C_0d_n \end{Bmatrix} U \quad \text{(III-69)}$$

where δ_1 and δ, the original variables of the equation, are related as shown previously, and U is the input variable to the transfer function expression as indicated in equation III-65.

b. m < n – The general expression for the case where m < n is easily accommodated by restricting the d_i coefficients to reflect the limit m. Commonly, only the d_0 coefficient will be finite.

4. Frequency Response

Transfer-function poles, zeros, and root gain can be converted to the standard Bode form for frequency response by combining time constants, damping, and resonant frequencies as

$$TF = k_B \frac{s^r \prod_{i=1}^{N1} (1 + \tau_i s) \prod_{i=1}^{N2} \left(1 + \frac{2\zeta_i s}{\omega_i} + \frac{s^2}{\omega_i^2}\right)}{\prod_{j=1}^{M1} (1 + \tau_j s) \prod_{j=1}^{M2} \left(1 + \frac{2\zeta_j s}{\omega_j} + \frac{s^2}{\omega_j^2}\right)} \qquad \text{(III-70)}$$

where the Bode gain is

$$k_B = k \frac{\prod_{i=1}^{n} z_i}{\prod_{j=1}^{m} p_j}$$

where

k = root gain

τ = system constants

ζ = system damping at frequency ω

ω = system resonant frequency

The frequency response is then calculated by substituting $j\omega$ for s and evaluating the transfer function expression at various ω's. The digital simulation uses a vernier frequency incrementing approach that automatically introduces smaller frequency increments near the poles

and zeros. This variable frequency incrementing technique permits better transfer-function resolution near the resonances at which amplitude and phase can vary rapidly.*

5. Root Locus

The root-locus method of analysis and design is based on the relationship between the poles and zeros of the closed-loop transfer function and those of the open-loop transfer function. The method is used to determine the location of the roots of the characteristic equation as a function of a single open-loop gain parameter. The locations of these roots are indicative of the relative system stability. The analyst may use the method as a design tool by adjusting the poles, zeros, and the open-loop gain parameters so as to yield a closed-loop system with satisfactory critical frequencies (poles and zeros).

To further describe the theoretical basis for the method, refer to the conventional control ratio for a feedback system as shown in figure 12.

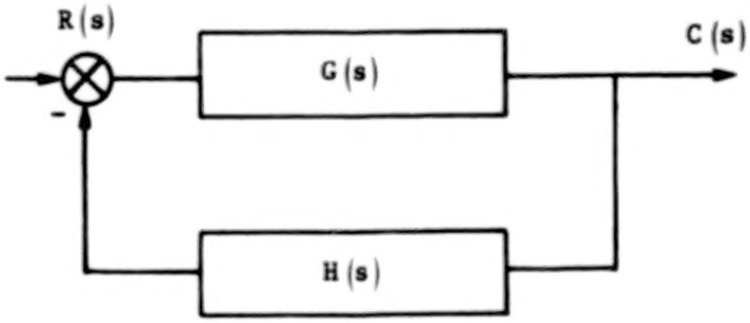


Figure 12. Conventional feedback control system.

The control ratio, C(s)/R(s), is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{III-71}$$

*Eigenvector analysis in conjunction with frequency domain analysis is not a common practice; however, for complex, highly coupled multirigid and flexible-body systems, it becomes a necessity. System roots often shift significantly from body and control frequencies, and are therefore impossible to physically interpret. Eigenvector analysis of the characteristic matrix associated with the transfer function under study can be used to determine a measure of the degree to which each of the body and control-system degrees of freedom couple to form the total system roots and associated modes (eigenvectors). For further information, see subroutine DEF5.

and the open-loop transfer function, G(s) H(s), is identified as a ratio of two functions in s,

$$G(s)H(s) = k\,\frac{P(s)}{Q(s)} \tag{III-72}$$

The characteristic system equation is

$$1 + G(s)H(s) = 0$$

or (III-73)

$$1 + k\,\frac{P(s)}{Q(s)} = 0$$

The conventional root-locus plot portrays the loci of the values of s that satisfy the characteristic equation as k varies from zero to infinity. Note that

- At k = 0, the roots of the characteristic equation are equal to the roots of Q(s), which are the same as the poles of the open-loop transfer function, k P(s)/Q(s).
- As k approaches infinity, the roots approach the roots of P(s), the open-loop zeros.

Thus, as k varies from 0 to infinity, the loci of the closed-loop poles migrate from the open-loop poles to the open-loop zeros, and the direction of migration depends on the sign of the open-loop gain parameter, k.

Rewriting equation III-73 yields a more conventional expression for the characteristic equation as

$$k\,\frac{P(s)}{Q(s)} = -1 \tag{III-74}$$

and two conditions are required as follows:

$$\left| k\,\frac{P(s)}{Q(s)} \right| = 1$$

$$\angle P(s)/Q(s) = 180^\circ,\ k \geqslant 0$$

The first of these conditions can be expressed as

$$k = \left| \frac{Q(s)}{P(s)} \right|$$

for those values of s that satisfy the angle criterion. The conditions that govern the migration of the roots in the complex plane can be solved by an iterative procedure. The iterative procedure for evaluating a single root locus* is described in Appendix E.

E. Linear Time-Domain Response

The linearized canonical first-order system of equations can also provide a basis for studying system time history in terms of perturbations about a specified state when the system indeed behaves in a linear manner in the vicinity of the state. The nonhomogeneous form of the equations, the basis for determining system transfer functions, appeared previously as

$$\dot{Z}^i = A_{ij} Z^j + b_{ik} U^k(t) \tag{III-75}$$

The external system inputs are the elements of U^k. It is convenient to establish the solution for the foregoing system through the use of a recursive-formula numerical-integration procedure rather than through the Runge-Kutta approach.

Consider the Adams' corrector formula (Reference 8) at time, t + 1,

$$\eta_{t+1} = \eta_t + \frac{h}{24}\left[9\,\dot{\eta}_{t+1} + 19\,\dot{\eta}_t - 5\,\dot{\eta}_{t-1} + \dot{\eta}_{t-2}\right] \tag{III-76}$$

where h is the incremented time step.

Application of this formula to our system of equations gives

$$z^i_{t+1} = z^i_t + \frac{h}{24}\left[9\,A_{ij}\,z^j_{t+1} + 9\,b_{ik}\,U^k_{t+1} + 19\,\dot{z}^i_t - 5\,\dot{z}^i_{t-1} + z^i_{t-2}\right]$$

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*Welch, Raymond V. NASA/Goddard Space Flight Center, Branch Report No. 254, October 2, 1973.

and manipulation yields the solution for all the z^i at time step, t + 1,

$$\{z\}_{t+1} = \left[[I] - \frac{3h}{8}[A]\right]^{-1} \left\{\{z\}_t + \frac{h}{24}\left(9\,[b]\,\{U\}_{t+1} + 19\,\{\dot{z}\}_t - 5\,\{\dot{z}\}_{t-1} + \{\dot{z}\}_{t-2}\right)\right\}$$

Note the requirement for z^i at time step, t - 2; hence, the requirement for a starter (e.g., Runge-Kutta) for initiating the solution process.

Goddard Space Flight Center
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Greenbelt, Maryland October 1977

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REFERENCE PAPER I

DYNAMIC RESPONSE AND STABILITY ANALYSIS OF FLEXIBLE MULTIBODY SYSTEMS (from DYNAMICS AND CONTROL OF LARGE FLEXIBLE SPACECRAFT, Proceedings of a Symposium held in Blacksburg, Virginia, June 13-15, 1977)

DYNAMIC RESPONSE AND STABILITY ANALYSIS OF FLEXIBLE, MULTIBODY SYSTEMS

by
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Abstract

An approach for dynamic simulation and stability analysis of systems of interconnected flexible bodies is discussed. The overall approach is unique in that any member body of the system can be flexible and the total system is not restricted to a topological tree configuration. The equations of motion are developed using the most general form of Lagrange's equations including auxiliary nonholonomic, rheonomic conditions of constraint. Lagrange multipliers are used as interaction forces/torques to maintain prescribed constraints. Nonlinear flexible/rigid dynamic coupling effects are accounted for in unabridged fashion for individual bodies and for the total system. Elastic deformation can be represented by normal vibration modes or by any adequate series of Rayleigh functions, including so-called quasi-static displacement functions.

A digital computer program system has been developed to numerically implement the modeling and analysis capability for a wide range of dynamic simulations including the nonlinear time domain and/or linear frequency domain. In particular, application has been made to: (1) time domain solution for nonlinear response of systems idealized as a collection of individual bodies; (2) numerical linearization of system governing equations; (3) time domain solution for perturbation response about a nominal state; and (4) frequency domain stability analysis corresponding to the linearized form.

Introduction

State of the art dynamic response analysis of a system of interconnected bodies, typical of current and projected large spacecraft, has